The numerical analysis of functional integral and integro-differential equations of Volterra type

Hermann Brunner
Department of Mathematics and Statistics,
Memorial University of Newfoundland,
St. John's, NL, Canada A1C 5S7
E-mail: hermann@math.mun.ca

The qualitative and quantitative analysis of numerical methods for delay differential equations is now quite well understood, as reflected in the recent monograph by Bellen and Zennaro (2003). This is in remarkable contrast to the situation in the numerical analysis of functional equations, in which delays occur in connection with memory terms described by Volterra integral operators. The complexity of the convergence and asymptotic stability analyses has its roots in new 'dimensions' not present in DDEs: the problems have distributed delays; kernels in the Volterra operators may be weakly singular; a second discretization step (approximation of the memory term by feasible quadrature processes) will in general be necessary before solution approximations can be computed.

The purpose of this review is to introduce the reader to functional integral and integro-differential equations of Volterra type and their discretization, focusing on collocation techniques; to describe the 'state of the art' in the numerical analysis of such problems; and to show that – especially for many 'classical' equations whose analysis dates back more than 100 years – we still have a long way to go before we reach a level of insight into their discretized versions to compare with that achieved for DDEs.

CONTENTS

1. Introduction

1.1. Early Volterra functional integral equations

1.1.1. Volterra integral equations with proportional delays

In his paper of 1897 (a sequel to his four fundamental papers that appeared in 1896), Vito Volterra studied the 'invertibility' of the 'definite integral' (using his notation)

$$f(y) - f(0) = \int_{\alpha y}^{y} \theta(x) H(x,y)\,\mathrm{d}x, \quad 0 < y < a, \tag{1.1}$$

where $0 < \alpha < 1$; the functions f, f', H, H_y are assumed to be continuous on their respective domains. The integral operator describing this first-kind integral equation has two variables of integration, and the lower limit represents a *proportional delay* vanishing at $t = 0$.

Volterra preceded the analysis of the existence and uniqueness of the solution $\theta \in C[0,a]$ by the following observation (Volterra 1897, pp. 156–157). Suppose that the given (real-valued) functions λ and φ are continuous on $[0,a]$, with $|\lambda(0)| \leq 1$, and consider the infinite series

$$\theta(x) := \varphi(x) + \sum_{j=1}^{\infty} \alpha^j \left(\prod_{l=0}^{j-1} \lambda(\alpha^l x) \right) \varphi(\alpha^j x), \quad x \in [0,a]. \tag{1.2}$$

This series converges uniformly, and hence its limit θ lies in $C[0,a]$. On the other hand, if $\theta \in C[0,a]$ is given, replacing x in (1.2) by αx and then multiplying by $\alpha\lambda(x)$ readily leads to an expression for the unknown φ,

$$\theta(x) - \alpha\lambda(x)\theta(\alpha x) = \varphi(x), \quad x \in [0,a]. \tag{1.3}$$

In other words, the pair of equations (1.2) and (1.3) are reciprocal to each other. This observation was then used by Volterra to establish the desired result for the delay integral equation (1.1) in a rather elegant way. We shall encounter (1.3) again later, as a special case of (2.12); see also Liu (1995*b*).

Volterra's analysis – which relies on Picard iteration techniques – was extended by Lalesco (1908, 1911) (see also Volterra (1913, pp. 92–101) and Fenyö and Stolle (1984, pp. 324–327)) to first-kind integral equations with more general vanishing delays, and by Andreoli (1913, 1914) to closely related integral equations of the second kind,

$$\varphi(x) + \lambda \int_0^{g(x)} N(x,y)\varphi(x)\,\mathrm{d}x = f(x), \quad x \in [0,a]. \tag{1.4}$$

Andreoli observed that '*la* $g(x)$ *avrà un'enorme influenza sulle formole di soluzione ...*' (the truth of this visionary remark regarding the analysis of discretized versions of such equations – especially when $g(x) = \alpha x$ $(0 < \alpha < 1)$ – will become apparent in Section 4.2!), and he illustrated it by means of two examples: $g(x) = \alpha x$ $(0 < \alpha < 1)$ and $g(x) = x^m$ $(m > 0;\ x \in [0,1])$.

1.1.2. The Volterra delay VIDEs of population dynamics

In Part IV ('Studio delle azioni ereditarie') of his 1927 paper Volterra refined his earlier celebrated (ODE) 'predator–prey' model to include situations where 'historical actions cease after a certain interval of time' (see also Volterra (1939, p. 8)). This leads to a system of nonlinear Volterra integro-differential equations with constant delay $T_0 > 0$ (again using Volterra's notation),

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = N_1(t)\left(\varepsilon_1 - \gamma_1 N_2(t) - \int_{t-T_0}^{t} F_1(t-\tau)N_1(\tau)\,\mathrm{d}\tau\right), \tag{1.5}$$

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2(t)\left(-\varepsilon_2 + \gamma_2 N_1(t) + \int_{t-T_0}^{t} F_2(t-\tau)N_2(\tau)\,\mathrm{d}\tau\right),$$

with $\varepsilon_i > 0$, $\gamma_i \geq 0$, and continuous $F_i(t) \geq 0$. Volterra later extended this model and its analysis to n interacting populations (see also his survey paper of 1939). Cushing (1977) is an excellent source on the further development of such population models based on VIDEs with delays; see also Bocharov and Rihan (2000) and its bibliography.

1.2. Volterra functional equations as mathematical models

Many basic mathematical models in epidemiology and population growth (Cooke and Yorke 1973, Waltman 1974, Cooke 1976, Smith 1977, Busenberg and Cooke 1980, Metz and Diekmann 1986 (especially Chapter IV), Hethcote and van den Driessche 2000, Brauer and van den Driessche 2003 (see also the extensive bibliographies in the last two papers)) are described by nonlinear Volterra integral equations of the second kind with (constant) delay $\tau > 0$:

$$y(t) = \int_{t-\tau}^{t} P(t-s)G(s, y(s))\,\mathrm{d}s + g(t), \quad t > t_0, \tag{1.6}$$

or

$$y(t) = \int_{t-\tau}^{t} P(t-s)G(y(s) + g(s))\,\mathrm{d}s, \quad t > t_0. \tag{1.7}$$

Here, g is usually assumed to be such that $\lim_{t\to\infty} g(t) =: g(\infty)$ exists. These delay integral equations model the deterministic growth of a population $y = y(t)$ (*e.g.*, of animals, or cells) or the spread of an epidemic with *immigration* into the population; it also has applications in economics.

A generalization of the above model is discussed in Bélair (1991): here, the delay τ in the delay (or lag) function $\theta(t) := t - \tau(y(t))$ (life span) is no longer constant but depends on the size $y(t)$ of the population at time t (reflecting, *e.g.*, crowding effects). Bélair's model corresponds to the delay VIE with *state-dependent delay*,

$$y(t) = \int_{t-\tau(y(t))}^{t} P(t-s)G(y(s))\,\mathrm{d}s, \quad t > 0, \tag{1.8}$$

with $P(t) \equiv 1$. Here it is assumed that the number of births is a function of the population size only (that is, the birth rate is density-dependent but not age-dependent). For this choice of the kernel P it is tempting to 'simplify' the delay VIE, by differentiating it with respect to t, to obtain the state-dependent (but 'local') DDE

$$y'(t) = \frac{G(y(t)) - G(y(t-\tau(y(t))))}{1 - \tau'(y(t))G(y(t-\tau(y(t))))}. \tag{1.9}$$

While *any* constant $y(t) = y_c$ solves the above DDE, this is *not* true in the original DVIE (1.8): it is easily verified that $y(t) = y_c$ is a solution if and only if $y_c = G(y_c)\tau(y_c)$. This simple example also contains a warning: the use of the the DDE (1.9) as the basis for the ('indirect') numerical solution of the delay VIE (1.8) may lead to approximations for $y(t)$ that do not correctly reflect the dynamics of the original (highly nonlinear) delay integral equation.

The elastic motions of a three-degree-of-freedom airfoil section with a flap in a two-dimensional incompressible flow can be described by a system of neutral functional integro-differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A_0 x(t) - \int_{-\tau}^{0} A_1(s)x(t+s)\,\mathrm{d}s\right) \tag{1.10}$$

$$= B_0 x(t) + B_1 x(t-\tau) + \int_{-\tau}^{0} K(s)x(t+s)\,\mathrm{d}s + F(t), \quad t > 0,$$

with $x(t) = \phi(t)$ $(-\tau \le t \le 0)$ and $\tau > 0$. Here, the matrices A_0, $A_1(\cdot)$, B_0, B_1 and $K(\cdot)$ in $L(\mathbb{R}^d)$ (with $d = 8$) are given. (Here, $L(\mathbb{R}^d)$ denotes the linear space of all real square matrices of order d.) The matrix A_0 is singular: typically, its last row consists of zeros, and some of the elements of $A_1(s) = (a_{ij}^{(1)}(s))$ are weakly singular, *e.g.*,

$$a_{88}^{(1)}(s) = C(s)(-s)^{-\alpha} + p(s), \quad 0 < \alpha < 1,$$

with smooth c and p. (See Burns, Cliff and Herdmann (1983*a*, 1983*b*), Burns, Herdman and Stech (1983*c*), Burns, Herdman and Turi (1987, 1990), and Herdman and Turi (1991) for details on the derivation and the mathematical framework of (1.10)).

The NFIDE (1.10) contains two new ingredients that make its analysis and the analysis of collocation methods significantly more difficult. The first complication is related to the occurrence of weakly singular kernel functions: they lead to solutions with unbounded derivatives at $t = 0^+$) and hence, on uniform meshes, to low order of convergence in collocation methods, regardless of the degree of the underlying piecewise polynomials. While there are ways to deal with this problem (compare Section 6.2 and, *e.g.*, Chapters 6 and 7 in Brunner (2004*b*)), it is not yet known how to overcome it when it occurs in conjunction with the (special) singular matrix A_0, since we are now facing a so-called *integro-differential algebraic system* (see März (2002*a*) for examples and a possible framework for their numerical analysis). For such problems (even when the kernel K is smooth) the analysis of numerical methods (based on a generalization of the notion of a *numerically properly formulated* DAE; see März (2002*a*, 2002*b*) and references) is very much in its infancy, but the subject of current joint work by R. Lamour, R. März, C. Tischendorf (Humboldt University, Berlin) and the author.

We conclude this section with a brief survey of the literature on applications of functional integral and integro-differential equations of Volterra type. Although this selection is necessarily subjective, taken together with the information contained in these books and papers (and their bibliographies) it will serve as a guide to the history and the present state of affairs of Volterra functional equations.

Starting with population dynamics (one of the major sources of Volterra integral and integro-differential equations with delay arguments) we mention the monographs by Volterra (1931), Volterra and d'Ancona (1935), Cushing (1977), Webb (1985), Brauer and Castillo-Chávez (2001), and Zhao (2003); the proceedings edited by Schmitt (1972), Metz and Diekmann (1986), and Ruan, Wolkowicz and Wu (2003); and the survey papers by Cooke and Yorke (1973), Busenberg and Cooke (1980), Ruan and Wu (1994), and Brauer and van den Driessche (2003). Among the milestone papers on this subject are the papers by Volterra (1927, 1928, 1934, 1939), Cooke (1976), Cooke and

Kaplan (1976), Smith (1977), Hethcote and Tudor (1980), Hethcote, Lewis and van den Driessche (1989), Cañada and Zertiti (1994), Hethcote and van den Driessche (1995, 2000). In addition, the reader may find it worthwhile to look at Tychonoff (1938) (for early applications of Volterra functional equations), Corduneanu and Lakshmikantham (1980) (on functional equations with unbounded delays), Ruan and Wu (1994) (on non-standard Volterra integro-differential equations), and Thieme and Zhao (2003), not least because of the numerous additional references contained in these papers.

Detailed treatments (and numerous additional applications) of nonlinear delay VIEs and VIDEs can be found in Marshall (1979), Lakshmikantham (1987), Györi and Ladas (1991), Yoshizawa and Kato (1991), Kolmanovskii and Myshkis (1992), Yatsenko (1995), Hritonenko and Yatsenko (1996), Piila (1996), Ruan and Wolkowicz (1996), and Corduneanu and Sandberg (2000). Compare also the papers by Tavernini (1978), and Cahlon and Nachman (1985), and their lists of references, on Volterra equations with state-dependent delays. The second chapter in Vogel (1965) contains an illuminating survey of the historical development of Volterra equations with delays and corresponding detailed references. Finally, the recent monograph by Ito and Kappel (2002) is the authoritative source for information on the mathematical framework for, and applications of, neutral functional integro-differential equations of the type (1.10).

2. Basic theory of Volterra functional integral equations I: non-vanishing delays

It goes without saying that a thorough understanding of the quantitative and qualitative properties of solutions to Volterra functional equations is essential for the design and the analysis of numerical methods for such problems. We therefore precede the sections dealing with the analysis of collocation methods (Sections 3 and 5) by brief sections giving an introduction to relevant theory of VFIEs, complemented by suggestions for additional readings. In this section we consider Volterra functional integral and integro-differential equations with *non-vanishing* delays.

The reader who – in order to see the subsequent analysis in a wider perspective – wishes to acquire a broader knowledge of the theory of delay differential equations is referred to, *e.g.*, the monographs by Myshkis (1972) (in Russian, with German translation of 1955), Bellman and Cooke (1963), El'sgol'ts and Norkin (1973), Kolmanovskii and Myshkis (1992), and – especially – Hale (1977), Hale and Verduyn Lunel (1993), Diekmann, van Gils, Verduyn Lunel and Walther (1995), and Wu (1996). Regularity results for solutions of DDEs may be found in Neves and Feldstein (1976), de Gee (1985), and Willé and Baker (1992).

2.1. Second-kind Volterra integral equations with non-vanishing delays

The general linear Volterra integral equation with delay $\theta(t)$ has the form

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T]. \tag{2.1}$$

Here, $\mathcal{V} : \ C(I) \to C(I)$ denotes the classical (linear) Volterra integral operator,

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} K_1(t,s)y(s)\,\mathrm{d}s, \tag{2.2}$$

with kernel $K_1 \in C(D)$, $D := \{(t,s) : \ t_0 \le s \le t \le T\}$. The kernel K_2 of the delay integral operator

$$(\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s \tag{2.3}$$

is assumed to be continuous in $D_\theta := \{(t,s) : \ \theta(t_0) \le s \le \theta(t), \ t \in I\}$, with $I := [t_0, T]$. Throughout this and the next section the delay (or lag) function θ will be subject to the following conditions:

(D1) $\theta(t) = t - \tau(t)$, with $\tau \in C^d(I)$ for some $d \ge 0$;

(D2) $\tau(t) \ge \tau_0 > 0$ for $t \in I$;

(D3) θ is strictly increasing on I.

We will refer to the function $\tau = \tau(t)$ as the delay.

Remark. The subsequent discussion will reveal that condition (D3) has been introduced mainly to simplify the description and the analysis of the collocation methods; the recent monograph by (Bellen and Zennaro 2003) on the numerical solution of DDEs deals with many of the complications that can arise if (D3) does not hold, providing illuminating examples and remarks.

We have seen in Section 1.2 that in applications (for example, in mathematical models for population growth or the spreading of an epidemic one encounters delay integral equations of the type

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{2.4}$$

corresponding to the delay integral operator

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)\,\mathrm{d}s, \tag{2.5}$$

or its nonlinear (Hammerstein) version,

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)G(s,y(s))\,\mathrm{d}s. \tag{2.6}$$

The (linear) delay equation (2.4) may be viewed as a particular case of (2.1), obtained formally by setting $K_2 = -K_1 =: -K$.

These delay integral equation are complemented, in analogy to DDEs, by an appropriate initial condition,

$$y(t) = \phi(t), \quad t \in [\theta(t_0), t_0].$$

We observe that, in contrast to initial-value problems for DDEs and DVIDEs with non-vanishing delays, the interval in which (2.1) and (2.4) are considered is the left-open interval $(t_0, T]$: we shall see below (Theorem 2.1) that solutions to Volterra integral equations with non-vanishing delays typically possess a finite (jump) discontinuity at $t = t_0$, while for first-order DDEs (and DVIDEs) the solution y is continuous at this point, with the discontinuity occurring in y'.

However, in complete analogy to DDEs the non-vanishing delay $\tau(\cdot)$ gives rise to the *primary discontinuity points* $\{\xi_\mu\}$ for the solution y: they are determined by the recursion

$$\theta(\xi_\mu) = \xi_\mu - \tau(\xi_\mu) = \xi_{\mu-1}, \quad \mu \geq 1, \quad \xi_\mu = t_0$$

(see, for example, Section 2.2 in Bellen and Zennaro (2003)). Condition (D2) ensures that these discontinuity points have the (uniform) separation property

$$\xi_\mu - \xi_{\mu-1} = \tau(\xi_\mu) \geq \tau_0 > 0, \quad \text{for all} \quad \mu \geq 1.$$

This implies that the number of primary discontinuity points in any bounded interval I remains finite: there is no clustering of the $\{\xi_\mu\}$.

Theorem 2.1. Assume that the given functions in (2.1)–(2.3) are continuous on their respective domains and that the delay function θ satisfies the above conditions (D1)–(D3). Then, for any initial function $\phi \in C[\theta(t_0), t_0]$, there exists a unique (bounded) $y \in C(t_0, T]$ solving the delay integral equation (2.1) on $(t_0, T]$ and coinciding with ϕ on $[\theta(t_0), t_0]$. In general, this solution has a finite (jump) discontinuity at $t = t_0$:

$$\lim_{t \to t_0^+} y(t) \neq \lim_{t \to t_0^-} y(t) = \phi(t_0).$$

The solution is continuous at $t = t_0$ if and only if the initial function is such that

$$g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0, s)\phi(s)\, \mathrm{d}s = \phi(t_0).$$

Proof. For $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ $(\mu \geq 1)$ the initial-value problem for (2.1) may be written as a Volterra integral equation of the second kind,

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^{t} K_1(t, s) y(s)\, \mathrm{d}s, \tag{2.7}$$

with $g_\mu(t) := g(t) + \Phi_\mu(t)$ and

$$\Phi_\mu(t) := \int_{t_0}^{\xi_\mu} K_1(t,s)y(s)\,\mathrm{d}s + \int_{t_0}^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s.$$

For $\mu = 0$ this function is known and given by

$$\Phi_0(t) = -\int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)\,\mathrm{d}s, \quad t \in I_0 := (t_0, \xi_1];$$

by our assumptions we have $\Phi_0 \in C(I^{(0)})$. It follows from the classical Volterra theory (Volterra (1896, 1913, 1959) and Miller (1971); see also Brunner and van der Houwen (1986), or Brunner (2004*b*)) that the integral equation (2.7) possesses a unique continuous (bounded) solution on each interval $I^{(\mu)}$ ($\mu \geq 0$).

As for its regularity, we first observe that for $\mu = 0$ (with $\xi_0 = t_0$),

$$\lim_{t \to t_0^+} y(t) = g(t_0) + \Phi_0(t_0) = g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0,s)\phi(s)\,\mathrm{d}s$$

which, for arbitrary (continuous) data g, K_2, ϕ, will not coincide with the value $\phi(t_0)$. For $\mu \geq 1$ we derive

$$y(\xi_\mu^-) = g(\xi_\mu) + \int_{t_0}^{\xi_\mu} K_1(\xi_\mu,s)y(s)\,\mathrm{d}s + \int_{t_0}^{\theta(\xi_\mu)} K_2(\xi_\mu,s)y(s)\,\mathrm{d}s$$

and

$$y(\xi_\mu^+) = g(\xi_\mu) + \int_{t_0}^{\xi_\mu} K_1(\xi_\mu,s)y(s)\,\mathrm{d}s + \int_{t_0}^{\theta(\xi_\mu)} K_2(\xi_\mu,s)y(s)\,\mathrm{d}s.$$

Hence,

$$y(\xi_\mu^+) - y(\xi_\mu^-) = 0,$$

whenever g, K_1, K_2 and θ are continuous functions. This completes the proof of Theorem 2.1. □

The solution of a linear Volterra integral equation of the second kind,

$$y(t) = g(t) + \int_{t_0}^{t} K(t,s)y(s)\,\mathrm{d}s, \quad t \in I,$$

with continuous g and K, can be expressed in term of the resolvent kernel $R = R(t,s)$ and the nonhomogeneous term g, namely,

$$y(t) = g(t) + \int_{t_0}^{t} R(t,s)g(s)\,\mathrm{d}s, \quad t \in I.$$

This 'variation-of-constants' formula is the key to establishing (global and local) superconvergence results for collocation solutions to such equations.

As the above proof implicitly shows, an analogous representation can be derived for the solution of the delay Volterra integral equation (2.1), since by (D2) the delay $\tau = \tau(t)$ in $\theta(t) = t - \tau(t)$ does not vanish in I. Suppose, for ease of notation and without loss of generality, that T in $I = [t_0, T]$ is such that $\xi_{M+1} = T$ (or, alternatively, $T \in (\xi_M, \xi_{M+1}])$ for some $M \geq 1$.

Theorem 2.2. Let (D1)–(D3) and the assumptions of Theorem 2.1 hold, and set

$$g_0(t) := g(t) - \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)\,\mathrm{d}s \quad \text{for} \quad t \in [t_0, \xi_1].$$

Then, for $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ ($\mu \geq 1$), the unique (bounded) solution y of (2.1) corresponding to the initial function ϕ can be expressed in the form

$$y(t) = g(t) + \int_{\xi_\mu}^{t} R_1(t,s)g(s)\,\mathrm{d}s + F_\mu(t) + \Phi_\mu(t), \tag{2.8}$$

with

$$F_\mu(t) := \int_{t_0}^{\xi_1} R_{\mu,0}(t,s)g_0(s)\,\mathrm{d}s + \sum_{\nu=1}^{\mu-1}\int_{\xi_\nu}^{\xi_{\nu+1}} R_{\mu,\nu}(t,s)g(s)\,\mathrm{d}s,$$

$$\Phi_\mu(t) := \int_{t_0}^{\theta^\mu(t)} Q_{\mu,0}(t,s)g_0(s)\,\mathrm{d}s + \sum_{\nu=1}^{\mu-1}\int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)g(s)\,\mathrm{d}s.$$

On the initial interval $I^{(0)} := (\xi_0, \xi_1]$ (with $\xi_0 = t_0$) the solution y is given by

$$y(t) = g_0(t) + \int_{t_0}^{t} R_1(t,s)g_0(s)\,\mathrm{d}s. \tag{2.9}$$

Here, R_1 is the resolvent kernel associated with the given kernel K_1 of the Volterra integral operator (2.2), $R_{\mu,\nu}$ and $Q_{\mu,\nu}$ denote functions which are continuous on their respective domains and depend on K_1, K_2, R_1 and θ, and $\theta^k := \underbrace{\theta \circ \cdots \circ \theta}_{k}$.

Remark. The structure of the above 'variation-of-constants' formula (2.8) and (2.9) clearly reveals the interaction between the classical lag term $F_\mu(t)$ (governed by the classical Volterra operator $\mathcal{V}$) and the delay term $\Phi_\mu(t)$ (which reflects the action of the non-vanishing lag function θ in $\mathcal{V}_\theta$). The structure of the latter will play a crucial role in the selection of appropriate ('θ-invariant') meshes for which local superconvergence results are possible (Sections 3.4.2 and 3.4.3).

Proof. The solution of the 'local' integral equation

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^{t} K_1(t,s)y(s)\,\mathrm{d}s, \quad t \in I^{(\mu)},$$

is given by

$$y(t) = g_\mu(t) + \int_{\xi_\mu}^{t} R_1(t,s)g_\mu(s)\,\mathrm{d}s, \quad t \in I^{(\mu)}, \tag{2.10}$$

with $R_1(t,s)$ defined by the resolvent equation

$$R_1(t,s) = K_1(t,s) + \int_{s}^{t} R_1(t,v)K_1(v,s)\,\mathrm{d}v, \quad (t,s) \in D^{(\mu)},$$

where $D^{(\mu)} := \{(t,s) : \ \xi_\mu \le s \le t \le \xi_{\mu+1}\}$. The expression (2.9) for the solution on the interval $I^{(0)}$ thus follows immediately.

On $I^{(1)}$ ($\mu = 1$) we thus have, using Dirichlet's formula,

$$\begin{aligned} g_1(t) &= g(t) + \int_{t_0}^{\xi_1} \left(K_1(t,s) + \int_{s}^{\xi_1} K_1(t,v)R_1(v,s)\,\mathrm{d}v \right) g_0(s)\,\mathrm{d}s \\ &\quad + \int_{t_0}^{\theta(t)} \left(K_2(t,s) + \int_{s}^{\theta(t)} K_2(t,v)R_1(v,s)\,\mathrm{d}v \right) g_0(s)\,\mathrm{d}s \\ &=: g(t) + \int_{t_0}^{\xi_1} Q_{1,1}^{(1)}(t,s)g_0(s)\,\mathrm{d}s + \int_{t_0}^{\theta(t)} Q_{1,0}^{(1)}(t,s)g_0(s)\,\mathrm{d}s, \end{aligned}$$

with obvious meaning of the (continuous) functions $Q_{1,0}^{(1)}$ and $Q_{1,1}^{(1)}$.

Recall now the representation (2.10) with $\mu = 1$ of the solution y on $I^{(1)}$: after trivial algebraic manipulation it can be written as

$$\begin{aligned} y(t) &= g(t) + \int_{t_0}^{t} R_1(t,s)g(s)\,\mathrm{d}s + \int_{t_0}^{\xi_1} \big(Q_{1,1}^{(1)}(t,s) + \hat{Q}_{1,1}^{(1)}(t,s)\big) g_0(s)\,\mathrm{d}s \\ &\quad + \int_{t_0}^{\theta(t)} \big(Q_{1,0}^{(1)}(t,s) + \hat{Q}_{1,0}^{(1)}(t,s)\big) g_0(s)\,\mathrm{d}s. \end{aligned}$$

This yields (2.8) with $\mu = 1$, by setting

$$R_{1,0}(t,s) := Q_{1,1}^{(1)}(t,s) + \hat{Q}_{1,1}^{(1)}(t,s), \qquad Q_{1,0}(t,s) := Q_{1,0}^{(1)}(t,s) + \hat{Q}_{1,0}^{(1)}(t,s).$$

Clearly, the functions describing this expression for y are continuous in the region where they are defined.

The proof is now concluded by a simple but notationally tedious induction argument. This argument reveals that, in the variation-of-constants formula (2.8), the integrals over $I^{(\mu)} = [\xi_\mu, \xi_{\mu+1}]$ with $\mu \ge 1$ will contribute terms involving only $g(t)$, while the integrals over $[\xi_0, \xi_1]$ and $[\xi_0, \theta^\mu(t)]$ contain the 'entire' initial function $g_0(t)$ which includes the contribution of the initial function ϕ. □

The result of Theorem 2.2 and its proof lead to the following result on the regularity of solutions of (2.1).

Theorem 2.3. Assume that (D1)–(D3) are satisfied, with $d \geq m \geq 1$ in (D1), and that the functions describing the delay Volterra integral equation (2.1) all possess continuous derivatives of at least order $m \geq 1$ on their respective domains. Then the following properties hold.

(a) The (unique) solution y of the initial-value problem for (2.1) is in $C^m(\xi_\mu, \xi_{\mu+1}]$ for each $\mu = 0, 1, \ldots, M$ and is bounded on $Z_M := \{\xi_\mu : \mu = 0, 1, \ldots, M\}$.

(b) At $t = \xi_\mu$ ($\mu = 1, \ldots, \min\{m, M\}$) we have

$$\lim_{t \to \xi_\mu^-} y^{(\mu-1)}(t) = \lim_{t \to \xi_\mu^+} y^{(\mu-1)}(t),$$

while the μth derivative of y is in general not continuous at ξ_μ. However, for $t \in [\xi_{m+1}, T]$ the solution lies in $C^m[\xi_{m+1}, T]$.

Remark. Differentiation of the Volterra delay integral equation of the *first kind*,

$$\int_{\theta(t)}^{t} H(t,s)y(s)\,\mathrm{d}s = f(t), \quad t \in I, \quad f(0) = 0, \tag{2.11}$$

leads – under appropriate regularity assumptions for H and f (see Section 2.2) – to a second-kind delay VIE that is somewhat more general than (2.1), namely

$$y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{W}_\theta y)(t), \quad t \in (\theta(t_0), t_0], \tag{2.12}$$

where we have set

$$g(t) := \frac{f'(t)}{K(t,t)}, \qquad b(t) := \frac{H(t,\theta(t))\theta'(t)}{H(t,t)},$$

and

$$K(t,s) := -\frac{\partial H(t,s)/\partial t}{H(t,t)}$$

in (2.12) and in the Volterra operator $\mathcal{W}_\theta$ (*cf.* (2.5)). Since the delay τ in $\theta(t) = t - \tau(t)$ does not vanish on I, the above result on the existence and uniqueness of a solution of the corresponding initial-value problem (Theorem 2.1), the variation-of-constants formula (Theorem 2.2), and the regularity properties (Theorem 2.3) can be generalized to encompass (2.11). We leave the proofs of these generalizations as an exercise. Note that for kernels with $\partial H(t,s)/\partial t \equiv 0$, equation (2.11) is closely related to (1.3).

For the sake of comparison we briefly consider the linear first-kind Volterra integral equation with delay function θ satisfying (D1)–(D3),

$$(\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t) = g(t), \quad t \in (t_0, T], \tag{2.13}$$

subject to the initial condition $y(t) = \phi(t)$, $t \in [\theta(t_0), t_0]$. The (linear) Volterra integral operators are those in (2.2) and (2.3).

Using the notation of the previous section we can write (2.13) in the local form

$$\int_{\xi_\mu}^{t} K_1(t,s)y(s)\,\mathrm{d}s = g_\mu(t), \quad t \in (\xi_\mu, \xi_{\mu+1}], \tag{2.14}$$

with

$$g_\mu(t) := g(t) - \int_{t_0}^{\xi_\mu} K_1(t,s)y(s)\,\mathrm{d}s - \int_{t_0}^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s, \tag{2.15}$$

for $\mu \geq 1$. For $t \in (\xi_0, \xi_1]$ this becomes

$$g_0(t) := g(t) + \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)\,\mathrm{d}s. \tag{2.16}$$

This reveals that for arbitrary continuous K_2, g, ϕ, θ, we have

$$g_0(t_0) = g(t_0) + \int_{\theta(t_0)}^{t_0} K_2(t_0,s)\phi(s)\,\mathrm{d}s \neq 0.$$

Hence, according to the classical Volterra theory of 1896, it follows that typically the solution of (2.13) (with $\mu = 0$) will be unbounded at $t = \xi_0^+ = t_0^+$:

$$\lim_{t \to t_0^-} y(t) = \phi(t_0) \neq \lim_{t \to t_0^+} y(t) = \pm\infty.$$

For the solution to be bounded at $t = t_0^+$ the initial function must be such that

$$\int_{\theta(t_0)}^{t_0} K_2(t_0,s)\phi(s)\,\mathrm{d}s = -g(t_0). \tag{2.17}$$

We summarize these observations in the following theorem.

Theorem 2.4. Assume:

(a) $K_1 \in C^1(D)$, with $|K_1(t,t)| \geq \kappa_0 > 0$, $t \in I := [t_0, T]$;
(b) $K_2 \in C^1(D_\theta)$;
(c) $g \in C^1(I)$;
(d) $\theta \in C^1$ is subject to (D1)–(D3) of Section 2.1, with $d = 1$ in (D1).

Then, for any $\phi \in C[\theta(t_0), t_0]$, there exists a unique y with $y \in C(\xi_\mu, \xi_{\mu+1}]$ ($\mu = 0, 1, \ldots, M$) which solves (2.13) on $(t_0, T]$ and coincides with ϕ on $[\theta(t_0), t_0]$. This solution y remains bounded at $t = t_0 = \xi_0$ if and only if (2.17) holds.

Is the smoothing property we encountered in solutions of delay Volterra integral equations of the second kind (Theorem 2.3) also present in solutions of the first-kind delay equation (2.13)? The simple but representative example

$$\int_{t_0}^{t} y(s)\,\mathrm{d}s + \int_{t_0}^{\theta(t)} \lambda_2 y(s)\,\mathrm{d}s = g(t), \quad t \in (t_0, T], \tag{2.18}$$

with $y(t) = \phi(t) = \phi_0$ for $t \in [\theta(t_0), t_0]$, whose solution can easily be found explicitly, shows that this is not so. The following theorem describes the general situation.

Theorem 2.5. Let the assumptions of Theorem 2.4 for the given functions in (2.13) hold, and assume that the initial function $\phi \in C[\theta(t_0), t_0]$ is such that the solution y of the initial-value problem for (2.13) is bounded at $t = t_0^+$ (*cf.* (2.17)). If y possesses a finite discontinuity at $t = t_0$, then it also has finite jumps at the other points of Z_M.

The extension of this regularity result to first-kind Volterra integral equations and to a class of related neutral functional integro-differential equations with weakly singular kernels will play an important role in the (not yet fully understood) analysis of convergence of collocation methods for such equations. See Brunner (1999*a*, 1999*b*) and the remarks in Section 6.3 below.

2.3. VIDEs with non-vanishing delays

We now turn to the (regularity) properties of solutions to the linear first-order delay VIDE

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [t_0, T], \tag{2.19}$$

corresponding to the Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ introduced in (2.2) and (2.3). It includes the analogue of the particular delay VIE (2.4),

$$y'(t) = a(t)y(t) + b(t)y(\theta(t)) + g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I, \tag{2.20}$$

with $\mathcal{W}_\theta$ given by (2.5) or (2.6).

The solutions y of the delay VIDE (2.19) (and hence those of (2.20)) will in general again have lower regularity at the *primary discontinuity points* $\{\xi_\mu\}$ defined by the recursion

$$\theta(\xi_\mu) = \xi_{\mu-1}, \quad \mu = 1, \ldots, \ (\xi_0 = t_0).$$

We start with a basic result on the existence and uniqueness of solutions of the initial-value problem for (2.19).

Theorem 2.6. Assume:

(a) $a,\ b,\ g,\ \theta \in C(I),\ K_1 \in C(D),\ K_2 \in C(D_\theta)$;
(b) $\theta(t) = t - \tau(t)$ satisfies the conditions (D1)–(D3) of Section 2.1.

Then, for any initial function $\phi \in C[\theta(t_0), t_0]$, there exists a unique function $y \in C(I) \cap C^1(t_0, T]$ which satisfies the delay VIDE (2.19) on I and coincides with ϕ on $[\theta(t_0), t_0]$. At $t = t_0$ its derivative is, in general, discontinuous (but bounded):

$$\lim_{t \to t_0^+} y'(t) \neq \lim_{t \to t_0^-} y'(t) = \phi'(t_0)$$

(assuming that $\theta'(t_0)$ exists).

The (unique) solution y of the initial-value problem for (2.19) can be expressed by a variation-of-constants formula, analogous to the one in Theorem 2.2 for the delay VIE (2.1). This result is based on the 'local' form of the above delay VIDE, that is, on the initial-value problem with respect to the interval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ $(\mu = 1, \ldots, M)$:

$$y'(t) = a(t)y(t) + g_\mu(t) + \int_{\xi_\mu}^{t} K_1(t,s)y(s)\,\mathrm{d}s, \quad t \in I^{(\mu)}, \tag{2.21}$$

where $y(\xi_\mu)$ is known and g_u is defined by

$$g_\mu(t) := g(t) + b(t)y(\theta(t)) + \int_{t_0}^{\xi_\mu} K_1(t,s)y(s)\,\mathrm{d}s + \int_{t_0}^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s. \tag{2.22}$$

For $\mu = 0$ the above lag term reduces to

$$g_0(t) := g(t) + b(t)\phi(\theta(t)) - \int_{\theta(t)}^{t_0} K_2(t,s)\phi(s)\,\mathrm{d}s, \quad t \in I^{(0)}. \tag{2.23}$$

The solution of the (local) VIDE (2.21) has the form

$$y(t) = r_1(t,\xi_\mu)y(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)g_\mu(s)\,\mathrm{d}s, \quad t \in I^{(\mu)}, \tag{2.24}$$

with the resolvent kernel r_1 given by the solution of the resolvent equation

$$\frac{\partial r_1(t,s)}{\partial s} = -r_1(t,s)a(s) - \int_{s}^{t} r_1(t,v)K_1(v,s)\,\mathrm{d}v, \quad (t,s) \in D^{(\mu)}, \tag{2.25}$$

subject to the initial condition $r_1(t,t) = 1$ for $t \in I^{(\mu)}$. (Compare also Grossman and Miller (1970), Brunner and van der Houwen (1986), or Brunner (2004*b*) for the theory of classical linear VIDEs.)

The following variation-of-constants formula is the analogue of the one presented in Theorem 2.2. Observe, however, that we now have additional terms involving the values of y at the primary discontinuity points $\{\xi_\mu\}$.

Theorem 2.7. Let the given functions a, b, g, K_1, K_2, ϕ be continuous, and assume that θ is subject to (D1)–(D3). Then on the interval $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ $(\mu \geq 1)$ the solution of the initial-value problem for (2.19) can be written as

$$y(t) = r_1(t,\xi_\mu)y(\xi_\mu) + \int_{\xi_\mu}^{t} r_1(t,s)g(s)\,\mathrm{d}s + F_\mu(t) + \Phi_\mu(t), \tag{2.26}$$

with

$$F_\mu(t) := \sum_{\nu=1}^{\mu-1} \rho_{\mu,\nu}(t)y(\xi_\nu) + \int_{\xi_0}^{\xi_1} r_{\mu,0}(t,s)g_0(s)\,\mathrm{d}s + \sum_{\nu=1}^{\mu-1}\int_{\xi_\nu}^{\xi_{\nu+1}} r_{\mu,\nu}(t,s)g(s)\,\mathrm{d}s,$$

$$\Phi_\mu(t) := \int_{\xi_0}^{\theta^\mu(t)} q_{\mu,0}(t,s)g_0(s)\,\mathrm{d}s + \sum_{\nu=1}^{\mu-1}\int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} q_{\mu,\nu}(t,s)g(s)\,\mathrm{d}s.$$

On the first interval $I^{(0)}$ this representation reduces to

$$y(t) = r_1(t,t_0)y(t_0) + \int_{t_0}^{t} r_1(t,s)g_0(s)\,\mathrm{d}s, \tag{2.27}$$

where $y(t_0) = \phi(t_0)$. The functions $\rho_{\mu,\nu}$, $r_{\mu,\nu}$, and $q_{\mu,\nu}$ depend on a, b, K_1, K_2, r_1 and θ and are continuous on their respective domains; $r_1 = r_1(t,s)$ denotes the resolvent kernel for $K_1 = K_1(t,s)$ defined by the resolvent equation (2.25).

Proof. The basic idea governing the proof of the above result is essentially the one used to establish Theorem 2.2, except that now the variation-of-constants formula is based on the resolvent representation of the solution of the 'local' VIDE (2.21) and will thus reflect the initial values $y(\xi_\mu)$. We leave the details of this simple proof to the reader. □

Remarks. **(1)** As in Theorem 2.2 we see again how the presence of the delay term $(\mathcal{V}_\theta y)(t)$ in (2.19) influences the resolvent representation of the classical (non-delay) VIDE on the macro-interval $I^{(\mu)}$. In addition, we now have terms reflecting the initial values $y(\xi_\nu)$ $(0 \leq \nu \leq \mu)$.

(2) There is a close connection between the representation of the solution of certain classes of Volterra functional (integro-)differential equations and the *semigroup framework* into which such equations can be embedded. Among the many papers dealing with this framework and corresponding solution representations we mention Burns *et al.* (1983*c*), Staffans (1985*a*, 1985*b*), Kappel and Zhang (1986), Burns, Herdman and Turi (1987), Clément, Desch and Homan (2002), and Ito and Kappel (2002).

2.4. Volterra functional equations with weakly singular kernels

As we have briefly seen in the remarks following equation (1.10), Volterra functional integral and integro-differential equations with weakly singular (*i.e.*, unbounded but integrable) kernels occur in many applications. Owing to limitations of space we shall not be able to say much about them in this paper, except to comment on open problems in their collocation analysis (Section 6.3). Here, we introduce relevant notation and point to papers in which the reader will find additional information.

Assume that $\alpha \in (0,1)$ is given, and define the delay integral operators

$$(\mathcal{V}_{\theta,\alpha} y)(t) := \int_0^{\theta(t)} (t-s)^{-\alpha} K(t,s) y(s)\,\mathrm{d}s, \tag{2.28}$$

and

$$(\mathcal{W}_{\theta,\alpha} y)(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha} K(t,s) y(s)\,\mathrm{d}s, \tag{2.29}$$

corresponding to continuous kernel functions K satisfying $|K(t,t)| \geq k_0 > 0$ when $s = t$. In Section 6.2 we will comment on some of the open problems arising for the corresponding functional equations

$$y(t) = g(t) + (\mathcal{T}_{\theta,\alpha} y)(t) \tag{2.30}$$

and

$$y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{T}_{\theta,\alpha} y)(t), \tag{2.31}$$

with $\mathcal{T}_{\theta,\alpha}$ representing one of the Volterra integral operators $\mathcal{V}_{\theta,\alpha}$ or $\mathcal{W}_{\theta,\alpha}$.

In Section 1.2 (equation (1.10)) we have encountered a (system of a) functional integro-differential equations of neutral type whose scalar counterpart may be written as (now using our standard notation),

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(a_0 y(t) - (\mathcal{W}_{\theta,\alpha} y)(t)\big) = F(t, y(t), y(\theta(t)), y'(\theta(t))), \quad t \geq 0,$$

with $0 < \alpha < 1$ and

$$(\mathcal{W}_{\theta,\alpha} y)(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha} K(t,s) y(s)\,\mathrm{d}s, \quad \theta(t) = t - \tau.$$

A related (but more complex) Volterra functional integro-differential equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big((\mathcal{W}_{\theta,\alpha} y)(t)\big) = f(t). \tag{2.32}$$

The mathematical analysis of functional equations of this type may be found in, *e.g.*, Kappel and Zhang (1986), Ito and Kappel (1991), Clément *et al.* (2002), and Ito and Kappel (2002). We will briefly return to these two classes of functional equations in Section 6.3. The mathematical (semigroup) framework for such equations has been developed in, *e.g.*, the papers and the monograph mentioned at the end of Section 2.3 (Remark (2)); results on the regularity of their solutions may be found in Brunner and Ma (2004).

3. Collocation methods for VFIEs with non-vanishing delays

3.1. Numerical analysis of VFIEs: an overview

We will use this section to sketch briefly the development of numerical methods for solving delay differential equations and more general functional integral and integro-differential equations of Volterra type. In the subsequent sections we shall then focus on collocation methods for such problems.

Most of the early discretization schemes for delay problems are based on 'classical' linear multistep and Runge–Kutta methods for ODEs. These methods have to be complemented by a suitable interpolation procedure (*e.g.*, by a *natural continuous extension* (NCE)), to generate approximations at certain non-mesh points $\theta(t)$. One of the principal merits of a collocation method is that the NCE is part of the method itself.

3.1.1. DDEs

The monograph by Myshkis (1972) (first published in Russian in 1955; see also the German translation of 1955) stands at the beginning of a sequence of distinguished monographs on the theory and applications of delay differential equations. Of these we mention Bellman and Cooke (1963), El'sgol'ts and Norkin (1973), Hale (1977), Kolmanovskii and Myshkis (1992), Hale and Verduyn Lunel (1993), Diekmann *et al.* (1995), Wu (1996) (on partial DDEs), and Ito and Kappel (2002) (on more general functional equations). The early survey papers by Halanay and Yorke (1971), Cryer (1972) and Bellen (1985), when read in 'hand-in-hand' with the recent ones by Zennaro (1995), Baker (1997, 2000), Baker and Paul (1997) and Bocharov and Rihan (2000), give a good idea of how the interest in theory, numerical analysis, and applications of DDE has grown since the early 1970s.

The monograph by Bellen and Zennaro (2003) provides not only a good introduction, by means of numerous illuminating examples, to the theory of DDEs but gives a state-of-the-art treatment of numerical methods for DDEs. Focusing on Runge–Kutta-type methods, we see that, beginning in the early 1980s, one can discern two main trends in the analysis of such methods. The first is concerned with the adaptation of (explicit and implicit) RK methods to DDEs and the construction of various interpolants, including NCEs.

Typical contributions are those by Bellen and Zennaro (1985), Zennaro (1986), in 't Hout (1992), and Vermiglio and Zennaro (1993) (Chapters 5 and 6 in Bellen and Zennaro (2003) contain a description of these quantitative aspects). Computational aspects are discussed in detail in Bellen and Zennaro (2003); compare also Neves and Thompson (1992) and Guglielmi and Hairer (2001*b*).

The second aspect is the study of asymptotic stability and contractivity properties of RK methods. Early milestones in the qualitative analysis of such methods are the papers by Reverdy (1981, 1990) and Torelli (1989). Of the many later contributions extending these results, the reader may also wish to consult those by Zennaro (1993, 1997), Torelli and Vermiglio (2003) Spijker (1997), Vermiglio and Torelli (1998), Guglielmi (1998) (dealing with delay-dependent stability), Guglielmi and Hairer (2001*a*) and Maset (2003). Finally, we mention the papers by Ascher and Petzold (1995) and Hauber (1997) on related numerical aspects for differential-algebraic equations with delays.

We shall see in Section 5.8 that the analogous analysis of the qualitative behaviour of collocation solutions for Volterra-type functional integral and integro-differential equations remains largely open.

Most of these papers consider only DDEs with constant delay $\tau > 0$. For general lag functions $\theta(t) = t - \tau(t)$ with *nonlinear* delay $\tau(t)$, the analysis and the implementation of IRK methods become much more complex (compare Lemma 3.1 below). This problem is not present in piecewise polynomial collocation methods since, as we have indicated before, they are global methods and thus automatically include an NCE. Bellen (1984) gave the first complete (super-) convergence analysis for such methods when applied to nonlinear DDEs with general nonlinear (non-vanishing) delays; his analysis is complemented in Vermiglio (1985). These collocation methods employ distinct collocation points; Hermite-type collocation for DDEs (and the attainable order of convergence) was studied by Oberle and Pesch (1981). More recent work on various aspects of collocation methods are studied in Enright and Hayashi (1998), Liu (1999*a*, 1999*b*), Engelborghs, Luzyanina, in 't Hout and Roose (2000), Engelborghs and Doedel (2002), and in Guglielmi and Hairer (2001*a*) (collocation at Radau II points).

We will not mention any of the superconvergence results here, since they can be obtained as particular cases of those for Volterra integro-differential equations with delays (see Section 3.1.3).

3.1.2. VFIEs of the second kind

As we saw in Section 1, the first papers on the theory of delay integral equations (Volterra (1897), Lalesco (1908, 1911), and Andreoli (1913, 1914)) considered the case of vanishing (proportional) delays. The development of the early theory for Volterra equations with non-vanishing delays is well sketched in Vogel (1965). Also of interest is the paper by Lin (1963), which contains a comparison result for solutions of systems of second-kind Volterra integral equations with constant delay. Additional results on the existence, uniqueness, and representation of solutions to such functional equations can be found in Levin and Nohel (1964), Bownds, Cushing and Schutte (1976), Cerha (1976) and Mureşan (1984, 1999), as well as in Cooke (1976), Esser (1976, 1978), Meis (1976), Busenberg and Cooke (1980), Cahlon, Nachman and Schmidt (1984) and Cañada and Zertiti (1994) (also for additional references). Chapter 4 in Brunner (2004*b*) contains an introduction to the theory of VIEs and VIDEs with non-vanishing delays.

The numerical analysis of Volterra integral equations with delays can be traced back to Esser (1976, 1978), Vâţă (1978), Wolff (1982), Cahlon *et al.* (1984) and Cahlon and Nachman (1985). More recent contributions on Runge–Kutta methods are those by Arndt and Baker (1988), Baker and Derakhshan (1990), Vermiglio (1992), as well as those by Cahlon (1990, 1992, 1995), Cahlon and Schmidt (1997), and Tian and Kuang (1995) (on the stability of numerical approximations). The reader may also wish to look at the survey papers by Cryer (1972) and Baker (1997, 2000).

Collocation methods in piecewise polynomial spaces occur in Vermiglio (1992), and their superconvergence properties are studied in detail in Brunner (1994*a*), Baddour and Brunner (1993), Hu (1997, 1999), and Brunner (2004*b*, Chapter 4).

3.1.3. VFIDEs

The literature on the theory and the numerical solution of VIDEs with delays is more extensive. It starts of course with Volterra's work (Volterra (1909, 1912) and, especially, (1927), (1931)). Of the numerous books we list the ones by Cushing (1977), Györi and Ladas (1991), Lakshmikantham, Wen and Zhang (1994), Ito and Kappel (2002), and Zhao (2003); see also the surveys by Corduneanu and Lakshmikantham (1980) and Jackiewicz and Kwapisz (1991), and their bibliographies. The regularity of solutions is analysed in, *e.g.*, Willé and Baker (1992) and in Brunner and Zhang (1999).

Important early contributions to the numerical solution of VFIDEs are due to Thompson (1968) and Tavernini (1971, 1973, 1978) (linear multistep and general one-step methods). We also mention the papers by Jackiewicz (1984), Arndt and Baker (1988), Jackiewicz and Kwapisz (1991), Makroglou (1983) (block methods for VIDEs with constant delay), Kazakova and Bainov (1990), Enright and Hu (1997) (continuous Runge–Kutta methods), and Baker and Tang (1997, 2000). Most of these methods are based on ODE schemes and hence they require an appropriate interpolation scheme to produce 'dense' data. The construction of NCEs for RK methods applied to classical VIDEs is the subject in Vermiglio (1988) (see also Bellen, Jackiewicz, Vermiglio and Zennaro (1989) for the case of delay VIEs); it can be extended to VIDEs with non-vanishing delays. Bellen (1985) and Baker (1997, 2000) contain comprehensive surveys and extensive lists of references on the numerical treatment of functional differential equations.

The numerical treatment of *partial* VIDEs with delay arguments have received increased attention in recent years. This topic is beyond the scope of this article (and the expertise of its author); the interested reader may consult Zubik-Kowal (1999) and Zubik-Kowal and Vandewalle (1999) for results and additional references.

Cryer and Tavernini (1972) study Euler's method for very general Volterra functional equations. This method may of course be interpreted as a simple collocation method. The (super-) convergence properties of piecewise polynomial collocation methods for delay VIDEs are described in Brunner (1994*b*), Burgstaller (1993, 2000), and Hu and Peng (1999); see also Chapter 4 in Brunner (2004*b*). The papers by Koto (2002) and by Brunner and Vermiglio (2003) investigate stability and contractivity properties of solutions to VIDEs with constant delays and neutral VFIDEs of 'Hale's form'. However, much work remains to be done before a good understanding of the qualitative (asymptotic) properties of collocation solutions to general (nonlinear) VFIDEs is obtained.

Finally, we mention another, important approach to the numerical solution of VFIDEs: it is based on a semigroup framework generated by the given functional equation (*cf.* also Clément *et al.* (2002) and references) and is able to deal with a rather general class of (linear) neutral VFIDEs. This approach originated in the work of Banks and Kappel (1979); see also Ito and Kappel (1989, 1991), Ito and Turi (1991), Clément *et al.* (2002), and, especially, the recent monograph by Ito and Kappel (2002).

3.2. Collocation methods for VFIEs with non-vanishing delays

In order to lead the reader not familiar with collocation methods for classical Volterra integral and integro-differential equations to their application to Volterra-type functional equations, we briefly summarize the principal ideas and mathematical tools underlying these global discretization methods.

3.2.1. Collocation spaces for classical Volterra equations

Let $I_h := \{t_n : \ 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a mesh on the interval $I := [0, T]$, and set

$$\sigma_n := (t_n, t_{n+1}], \quad \bar{\sigma}_n := [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n \ \ (0 \le n \le N-1);$$

the diameter of the mesh I_h is $h := \max_{(n)} h_n$. For given integers $m \ge 1$ and $d \ge -1$ we let

$$S^{(d)}_{m+d}(I_h) := \{u_h \in C^d(I) : \ u_h|_{\sigma_n} \in \pi_{m+d} \ (0 \le n \le N-1)\} \tag{3.1}$$

denote the linear space of (real) piecewise polynomials with respect to the mesh I_h whose degree does not exceed $m+d$. If $d=-1$ then $u_h \in S_{m-1}^{(-1)}(I_h)$ will in general have finite (jump) discontinuities at the interior points of I_h; the space of step functions, $S_0^{(-1)}(I_h)$, is the most obvious example of such a discontinuous piecewise polynomial space.

The dimension of the linear space defined by (3.1) is given by

$$\dim S_{m+d}^{(d)}(I_h) = Nm + (d+1).$$

The choice of d, the degree of regularity, will be governed by the type of functional equation whose solution will be approximated by collocation in the linear space $S_{m+d}^{(d)}(I_h)$: for the functional integral equations not containing derivatives of the unknown solution the 'natural' piecewise polynomial space is $S_{m-1}^{(-1)}(I_h)$ $(d=-1)$, while for functional integro-differential equations in which the highest derivative of the unknown solution is $y^{(k)}$ $(k \geq 1)$ we choose $d = k-1$.

The desired collocation solution $u_h \in S_{m+d}^{(d)}(I_h)$ will be determined by requiring that it satisfy the given functional equation on the set of collocation points

$$X_h := \{t_{n,i} := t_n + c_i h_n :\ 0 < c_1 < \cdots < c_m \leq 1\ (0 \leq n \leq N-1)\}, \quad (3.2)$$

described by given *collocation parameters* $\{c_i\}$. Clearly,

$$\dim S_{m+d}^{(d)}(I_h) = Nm + (d+1) = |X_h| + (d+1).$$

If $d \geq 0$ the collocation solution will also be required to coincide, at $t=0$, with the prescribed initial value(s); *e.g.*, in the case of the DVIDEs (2.19) and (2.20) $(k=1)$ we have $u_h(0) = y_0$.

3.2.2. Constrained and θ-invariant meshes

Assume that the given lag function $\theta(t) = t - \tau(t)$ satisfies the assumptions (D1)–(D3) of Section 2.1, which we will recall for the convenience of the reader:

(D1) $\theta \in C^d(I)$ for some $d \geq 0$, with $I := [t_0, T]$;
(D2) $\tau(t) \geq \tau_0 > 0$ for $t \in I$;
(D3) θ is strictly increasing on I.

We have seen, in the comments preceding Theorem 2.1, that the primary discontinuity points $\{\xi_\mu\}$, induced by θ and given by $\theta(\xi_\mu) = \xi_{\mu-1}$ $(\mu = 1, \cdots;\ \xi_0 := t_0)$, possess the (uniform) separation property $\xi_\mu - \xi_{\mu-1} \geq \tau_0 > 0$ for all $\mu \geq 1$. For ease of notation we will again assume that T defining $I = [t_0, T]$ is such that

$$T = \xi_{M+1} \quad \text{for some} \quad M \geq 1,$$

and we set $Z_M := \{\xi_\mu : \ \mu = 0, 1, \ldots, M\}$.

Since solutions of delay problems with non-vanishing delays generally suffer from a loss of reguarity at the primary discontinuity points $\{\xi_\mu\}$, the mesh I_h underlying the collocation space will have to include these points if the collocation solution is to attain its optimal global (or local) order (of superconvergence. Thus, we shall employ meshes of the form

$$I_h := \bigcup_{\mu=0}^{M} I_h^{(\mu)}, \quad I_h^{(\mu)} := \{t_n^{(\mu)} : \ \xi_\mu = t_0^{(\mu)} < t_1^{(\mu)} < \cdots < t_{N_\mu}^{(\mu)} = \xi_{\mu+1}\}. \quad (3.3)$$

Such a mesh is called a *constrained mesh* (with respect to θ) for I. We will refer to I_h as the *macro-mesh* and call the $I_h^{(\mu)}$ the underlying *local meshes*.

Definition. A mesh I_h for $I := [t_0, T]$ is said to be θ-*invariant* if it is constrained (that is, given by (3.3)) and if

$$\theta(I_h^{(\mu)}) = I_h^{(\mu-1)}, \quad \mu = 1, \ldots, M. \quad (3.4)$$

We then have $N_\mu = N$ for all $\mu \geq 0$.

Observe that if I_h is θ-invariant then

$$t \in I_h^{(\mu)} \Longrightarrow \theta^{\mu-\nu}(t) \in I_h^{(\nu)}, \quad \nu = 0, 1, \ldots, \mu. \quad (3.5)$$

In analogy to Section 3.2.1 we will use the following notation:

$$\sigma_n^{(\mu)} := (t_n^{(\mu)}, t_{n+1}^{(\mu)}], \quad h_n^{(\mu)} := t_{n+1}^{(\mu)} - t_n^{(\mu)}, \quad h^{(\mu)} := \max_{(n)} h_n^{(\mu)}, \quad h := \max_{(\mu)} h^{(\mu)},$$

and $\bar{\sigma}_n^{(\mu)} := [t_n^{(\mu)}, t_{n+1}^{(\mu)}]$.

For a given θ-invariant mesh I_h the collocation solution u_h will be an element of a piecewise polynomial space

$$S_{m+d}^{(d)}(I_h) := \{v \in C^d(I_h) : \ v|_{\sigma_n^{(\mu)}} \in \pi_{m+d} \ (0 \leq n < N; \ 0 \leq \mu \leq M)\}. \quad (3.6)$$

It follows from Section 3.2.1 that this linear space has the dimension

$$\dim S_{m+d}^{(d)}(I_h) = (M+1)Nm + d + 1.$$

Hence the collocation points will now be chosen as

$$X_h := \bigcup_{\mu=0}^{M} X_h^{(\mu)}; \quad (3.7)$$

they are based on the $M+1$ local sets

$$X_h^{(\mu)} := \{t_n^{(\mu)} + c_i h_n^{(\mu)} : \ 0 < c_1 < \cdots < c_m \leq 1 \ (0 \leq n \leq N-1)\}$$

of cardinality Nm. In the collocation equation for a given delay equation with *non-vanishing* delay $\tau(t)$, we shall encounter the mapping $\theta(X_h^{(\mu)})$

(see, for example, (3.9) below). It is clear that for *linear lag functions* θ and given θ-invariant mesh I_h the set X_h defined by (3.7) is also θ-invariant. However, for *nonlinear* delays this will no longer be true. We record this important fact – which will affect the computational form of the collocation equation – in the following lemma. Its proof is straightforward and is left as an exercise.

Lemma 3.1. Assume that the delay function θ satisfies (D1)–(D3), and let I_h be a θ-invariant mesh on $I = [t_0, T]$.

(a) If θ is linear, then

$$\theta(X_h^{(\mu)}) = X_h^{(\mu-1)}, \quad \mu = 1, \ldots, M,$$

and the set X_h of collocation points is also θ-invariant.

(b) For nonlinear θ this is no longer true: setting

$$\theta(t_n^{(\mu)} + c_i h_n^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} =: \tilde{t}_{n,i}^{(\mu-1)}, \quad i = 1, \ldots, m,$$

the images $\{\tilde{c}_i\}$ of the $\{c_i\}$ satisfy

$$0 \le \tilde{c}_1 < \cdots < \tilde{c}_m \le 1 \quad (\text{with } \ \tilde{c}_i \ne c_i \ \text{ in general}),$$

and they depend on the micro-interval $\sigma_n^{(\mu)}$ and the macro-interval $I^{(\mu)}$:

$$\tilde{c}_i = \tilde{c}_i(n; \mu), \quad i = 1, \ldots, m.$$

3.3. Delay integral equations of the second kind

3.3.1. The collocation equations

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the delay integral equation

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T], \tag{3.8}$$

with

$$(\mathcal{V}y)(t) := \int_{t_0}^{t} K_1(t,s)y(s)\,\mathrm{d}s, \qquad (\mathcal{V}_\theta y)(t) := \int_{t_0}^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s,$$

and with initial condition $y(t) = \phi(t)$, $t \le t_0$, is defined by the collocation equation

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h. \tag{3.9}$$

The values of u_h at $t \in [\theta(t_0), t_0]$ are determined by the given initial function for (3.8), $u_h(t) = \phi(t)$. As for classical second-kind Volterra integral equations we will also consider the *iterated collocation solution* corresponding to u_h:

$$u_h^{\mathrm{it}}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in (t_0, T]. \tag{3.10}$$

The lag function $\theta = \theta(t) = t - \tau(t)$ will be assumed to satisfy the conditions (D1)–(D3) of Section 3.2.2, and the mesh I_h on $I := [t_0, T]$ will be assumed to be the θ-invariant mesh defined by (3.3) and (3.4).

On $\sigma_n^{(\mu)} := (t_n^{(\mu)}, t_{n+1}^{(\mu)}]$ the collocation solution will have the usual local Lagrange representation,

$$u_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) U_{n,j}^{(\mu)}, \quad v \in (0,1], \quad \text{with} \quad U_{n,j}^{(\mu)} := u_h(t_{n,j}^{(\mu)}). \tag{3.11}$$

Since the contribution of the classical Volterra term $\mathcal{V}u_h$ to the computational form of the collocation equation is obvious, we will focus here on the terms induced by the delay part $(\mathcal{V}_\theta u_h)(t)$ with $t = t_{n,i}^{(\mu)}$.

Assume first that the delay θ is *linear*. As we have seen in Lemma 3.1, the θ-invariance of the mesh I_h implies the θ-invariance of the set X_h of collocation points; thus we may write, using the fact that $\theta(t_{n,i}^{(\mu)}) = t_{n,i}^{(\mu-1)}$,

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) &= \int_{t_0}^{\theta(t_{n,i}^{(\mu)})} K_2(t_{n,i}^{(\mu)}, s) u_h(s)\, \mathrm{d}s \\ &= \int_{t_0}^{t_{n,i}^{(\mu-1)}} K_2(t_{n,i}^{(\mu)}, s) u_h(s)\, \mathrm{d}s, \end{aligned} \tag{3.12}$$

and hence, recalling the local representation (3.11) of u_h,

$$\begin{aligned} (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) &= \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) \\ &\quad + h_n^{(\mu-1)} \sum_{j=1}^{m} \Big(\int_0^{c_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s)\, \mathrm{d}s \Big) U_{n,j}^{(\mu-1)}, \end{aligned} \tag{3.13}$$

with lag term

$$\begin{aligned} \Psi_n^{(\mu-1)}(t) &:= \int_{t_0}^{\xi_{\mu-1}} K_2(t,s) u_h(s)\, \mathrm{d}s \\ &\quad + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} K_2(t,s) u_h(s)\, \mathrm{d}s, \quad t \in \sigma_n^{(\mu)}. \end{aligned} \tag{3.14}$$

If the delay θ is *nonlinear*, then the above terms have to be modified: by the (strict) monotonicity assumption (D3) for θ the image of $t_{n,i}^{(\mu)} \in \sigma_n^{(\mu)}$ under θ lies in $\sigma_n^{(\mu-1)}$ (but will be different from the collocation point $t_{n,i}^{(\mu-1)} = t_n^{(\mu-1)} + c_i h_n^{(\mu-1)}$); that is,

$$\theta(t_{n,i}^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)} =: \tilde{t}_{n,i}^{(\mu-1)}, \quad i = 1, \dots, m, \tag{3.15}$$

with

$$0 < \tilde{c}_1 < \cdots < \tilde{c}_m \leq 1 \quad \text{and} \quad \tilde{c}_i = \tilde{c}_i(n;\mu)$$

(*cf.* Lemma 3.1). Accordingly, the expression (3.13) for $(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)})$ now reads

$$(\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}) = \Psi_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu-1)} \sum_{j=1}^{m} \Big(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s)\, \mathrm{d}s \Big) U_{n,j}^{(\mu-1)}. \tag{3.16}$$

Hence, the collocation equation (3.9) at $t = t_{n,i}^{(\mu)}$ $(i = 1, \ldots, m)$ can now be written as

$$U_{n,i}^{(\mu)} = h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s)\, \mathrm{d}s \Big) U_{n,j}^{(\mu)} + g(t_{n,i}^{(\mu)}) + F_n^{(\mu)}(t_{n,i}^{(\mu)}) + (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}). \tag{3.17}$$

The classical lag term (corresponding to the Volterra operator $\mathcal{V}$ in (3.9)) has, for $t \in \sigma_n^{(\mu)}$, the form

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} K_1(t,s) u_h(s)\, \mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t,s) u_h(s)\, \mathrm{d}s. \tag{3.18}$$

Let $\mathbf{U}_n^{(\mu)} := (U_{n,1}^{(\mu)}, \ldots, U_{n,m}^{(\mu)})^T \in \mathbb{R}^m$ and define the matrices in $L(\mathbb{R}^m)$,

$$B_n^{(\mu)} := \Big(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) L_j(s)\, \mathrm{d}s \Big)_{i,j=1}^{m},$$

$$\tilde{B}_n^{(\mu-1)} := \Big(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) L_j(s)\, \mathrm{d}s \Big)_{i,j=1}^{m}.$$

Finally, set

$$\mathbf{g}_n^{(\mu)} := (g(t_{n,1}^{(\mu)}), \ldots, g(t_{n,m}^{(\mu)}))^T,$$

$$\mathbf{G}_n^{(\mu)} := (F(t_{n,1}^{(\mu)}), \ldots, F_n^{(\mu)}(t_{n,m}^{(\mu)}))^T,$$

and

$$\mathbf{Q}_n^{(\mu-1)} := (\Psi_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \ldots, \Psi_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T.$$

Thus, the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (3.8) on $\sigma_n^{(\mu)}$ is described by (3.11), in which $\mathbf{U}_n^{(\mu)}$ is the solution of the linear algebraic system

$$[I_m - h_n^{(\mu)} B_n^{(\mu)}]\mathbf{U}_n^{(\mu)} = \mathbf{g}_n^{(\mu)} + \mathbf{G}_n^{(\mu)} + \mathbf{Q}_n^{(\mu-1)} + h_n^{(\mu-1)} \tilde{B}_n^{(\mu)} \mathbf{U}_n^{(\mu-1)}, \tag{3.19}$$

where $n = 0, 1, \ldots, m$ and $\mu = 0, 1, \ldots, M$. The matrix I_m denotes the identity operator in $L(\mathbb{R}^m)$.

The following theorem on the existence of a unique collocation solution is an obvious consequence of the uniform boundedness of the inverses of the matrices $\mathcal{B}_n^{(\mu)} := I_m - h_n^{(\mu)} B_n^{(\mu)}$ for sufficiently small mesh diameters h.

Theorem 3.2. Assume that g, θ, K_1 and K_2 are continuous on their respective domains I, D and D_θ, with the lag function θ satisfying (D1)–(D3).

Then there exists an $\bar{h} > 0$ such that, for any θ-invariant mesh I_h with $h \in (0, \bar{h})$ and any initial function $\phi \in [\theta(t_0), t_0]$, each of the linear algebraic systems (3.19) possesses a unique solution $\mathbf{U}_n^{(\mu)} \in \mathbb{R}^m$. Hence, the collocation equation (3.9) defines a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (3.8) whose local representation on the subintervals $\sigma_n^{(\mu)}$ is given by (3.11).

The computational form of the *iterated collocation solution* (3.10) at $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ can be written as

$$\begin{aligned} u_h^{\text{it}}(t) &= g(t) + F_n^{(\mu)}(t) + \Psi_n^{(\mu-1)}(t) \\ &\quad + h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^v K_1(t, t_n^{(\mu)} + sh_n^{(\mu)}) L_j(s)\, \mathrm{d}s \Big) U_{n,j}^{(\mu)} \\ &\quad + h_n^{(\mu-1)} \sum_{j=1}^{m} \Big(\int_0^{\tilde{v}} K_2(t, t_n^{(\mu-1)} + sh_n^{(\mu-1)}) L_j(s)\, \mathrm{d}s \Big) U_{n,j}^{(\mu-1)}. \end{aligned} \tag{3.20}$$

Recall that the lag term $\Psi_n^{(\mu-1)}(t)$ corresponding to the delay operator $\mathcal{V}_\theta$ is given above by (3.14). The image $\tilde{t} := t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}$ of $t = t_n^{(\mu)} + vh_n^{(\mu)}$ under θ depends on the nature of the lag function θ: if θ is *linear* then we have $\tilde{v} = v$; for *nonlinear* θ the value of $\tilde{v} \in [0, 1]$ must be obtained from

$$\theta(t_n^{(\mu)} + vh_n^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}, \quad v \in (0, 1]. \tag{3.21}$$

We note in passing that $u_h^{\text{it}} \in C[t_0, T]$ whenever the given data defining the initial-value problem for (3.8) are continuous functions and we have

$$u_h^{\text{it}}(t_0) = g(t_0) - \int_{\theta(t_0)}^{t_0} K_2(t_0, s)\phi(s))\, \mathrm{d}s \ (= y(t_0^+)).$$

Moreover,

$$u_h^{\text{it}}(t) = u_h(t) \quad \text{for all} \ \ t \in X_h.$$

Since second-kind Volterra integral equations with non-vanishing delays often arise in the particular form

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{3.22}$$

where

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} K(t,s)y(s)\,\mathrm{d}s,$$

we present the corresponding computational form of the collocation equation defining $u_h \in S_{m-1}^{(-1)}(I_h)$ in some detail (although it could of course be formally obtained by setting $K_2 = -K_1 =: -K$ in (3.17). We first note that for $t = t_{n,i}^{(\mu)}$ we have

$$\begin{aligned}(\mathcal{W}_\theta u_h)(t) = &\int_{\theta(t)}^{t_{n+1}^{(\mu-1)}} K(t,s)u_h(s)\,\mathrm{d}s \qquad (3.23)\\ &+ \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} K(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} K(t,s)u_h(s)\,\mathrm{d}s\\ &+ h_n^{(\mu)} \int_0^{c_i} K(t, t_n^{(\mu)} + sh_n^{(\mu)})u_h(t_n^{(\mu)} + sh_n^{(\mu)})\,\mathrm{d}s,\end{aligned}$$

where

$$\theta(t) = \theta(t_{n,i}^{(\mu)}) = \begin{cases} t_{n,i}^{(\mu-1)} = t_n^{(\mu-1)} + c_i h_n^{(\mu-1)}, & \text{if } \theta \text{ is } \textit{linear},\\ \tilde{t}_{n,i}^{(\mu-1)} := t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)}, & \text{if } \theta \text{ is } \textit{nonlinear}.\end{cases}$$

Define, for $t = t_n^{(\mu)} + c_i h_n^{(\mu)}$,

$$\begin{aligned}\bar{\Psi}_n^{(\mu-1)}(t) := h_n^{(\mu-1)} &\int_{\tilde{c}_i}^{1} K(t, t_n^{(\mu-1)} + sh_n^{(\mu-1)})u_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)})\,\mathrm{d}s\\ &+ \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} K(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} K(t,s)u_h(s)\,\mathrm{d}s. \qquad (3.24)\end{aligned}$$

The collocation equation for (3.22) on $\sigma_n^{(\mu)}$ then becomes

$$\begin{aligned}U_{n,i}^{(\mu)} = &\, g(t_{n,i}^{(\mu)}) + \bar{\Psi}_n^{(\mu-1)}(t_{n,i}^{(\mu)}) \qquad (3.25)\\ &+ h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^{c_i} K(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)})L_j(s)\,\mathrm{d}s \Big) U_{n,j}^{(\mu)}, \quad i = 1,\ldots,m.\end{aligned}$$

Hence, the resulting linear algebraic system for $U_n^{(\mu)} \in \mathbb{R}^m$ defining the local representation of u_h on $\sigma_n^{(\mu)}$ (*cf.* (3.11)) has the form

$$[I_m - h_n^{(\mu)} B_n^{(\mu)}]\mathbf{U}_n^{(\mu)} = \mathbf{g}_n^{(\mu)} + \bar{\mathbf{G}}_n^{(\mu-1)}, \qquad (3.26)$$

with

$$\mathbf{g}_n^{(\mu)} := (g(t_{n,1}^{(\mu)}, \ldots, g(t_{n,m}^{(\mu)})^T$$

and

$$\bar{\mathbf{G}}_n^{(\mu-1)} := (\bar{\Psi}_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \ldots, \bar{\Psi}_n^{(\mu-1)}(t_{n,m}^{(\mu)}))^T.$$

The corresponding *iterated collocation solution* at $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ can then be computed via

$$\begin{aligned} u_h^{\text{it}}(t) &= g(t) + \bar{\Psi}_n^{(\mu-1)}(t) \\ &\quad + h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^{c_i} K(t, t_n^{(\mu)} + sh_n^{(\mu)}) L_j(s)\,\mathrm{d}s \Big) U_{n,j}^{(\mu)}. \end{aligned} \tag{3.27}$$

3.3.2. Global convergence results

The collocation error $e_h := y - u_h$ associated with the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the delay integral equation (3.8) solves the initial-value problem

$$e_h(t) = \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in (t_0, T], \tag{3.28}$$

with initial condition $e_h(t) = 0$ for $t \in [\theta(t_0), t_0]$. The defect δ_h, defined by

$$\delta_h(t) := -u_h(t) + g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I,$$

vanishes on the set X_h. For $t \in \sigma_n^{(\mu)}$ $(\mu \geq 1)$ the above error equation can be written as

$$e_h(t) = E_\mu(t) + \delta_h(t) + \int_{\xi_\mu}^{t} K_1(t,s) e_h(s)\,\mathrm{d}s, \tag{3.29}$$

where

$$E_\mu(t) := \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} K_1(t,s) e_h(s)\,\mathrm{d}s + (\mathcal{V}_\theta e_h)(t). \tag{3.30}$$

On the first macro-interval $(t_0, \xi_1]$ we have

$$E_0(t) := (V_\theta e_h)(t) = -\int_{\theta(t)}^{t_0} K_2(t,s) e_h(s)\,\mathrm{d}s = 0.$$

If the given functions in (3.8) have continuous derivatives of at least order m on their respective domains, the global convergence and order analysis can be based on the (local) representation of the collocation error based on the Peano Kernel Theorem for polynomial interpolation. This representation has the form

$$e_h(t_n^{(\mu)} + vh_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) \mathcal{E}_{n,j}^{(\mu)} + (h_n^{(\mu)})^m R_{m,n}^{(\mu)}(v), \quad v \in (0,1], \tag{3.31}$$

with $\mathcal{E}_{n,j}^{(\mu)} := e_h(t_{n,j}^{(\mu)})$ and Peano remainder term $R_{m,n}^{(\mu)}(v)$ (see Brunner (2004*b*, Chapters 1 and 2) for details). On the first macro-interval $[\xi_0, \xi_1]$

the estimate for e_h is the one for classical Volterra integral equations of the second kind (Brunner and van der Houwen 1986, Chapter 5):

$$\|e_h\|_{0,\infty} := \sup_{t\in I^{(0)}} |e_h(t)| \le C_0(h^{(0)})^m, \quad n = 0, 1, \dots, N-1;$$

it is a consequence of the estimate $\|\boldsymbol{\mathcal{E}}_n^{(0)}\|_1 = \mathcal{O}((h^{(0)})^m)$ (where $\boldsymbol{\mathcal{E}}_n^{(\mu)} := (\mathcal{E}_{n,1}^{(\mu)}, \dots, \mathcal{E}_{n,m}^{(\mu)})^T$). A simple induction argument, employing the estimates for the terms $E_\mu(t)$ ($t \in I^{(\mu)}$) in (3.29) and (3.30), together with the observation that by the conditions (D1)–(D3) for the delay θ the number $(M+1)$ of macro-intervals $I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ is finite, yields the results summarized in the following theorem.

Theorem 3.3. Let the following conditions be satisfied.

(a) The given functions g, K_1, K_2 and ϕ in (3.8) all possess continuous derivatives of order m on their respective domains.

(b) The delay function $\theta(t) = t - \tau(t)$ is subject to the conditions (D1)–(D3) of Section 3.2.2, with $d \ge m$ in (D1).

(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution to (4.8) corresponding to a θ-invariant mesh I_h with $h \in (0, \bar{h}$, where $\bar{h}$ is defined in Theorem 3.2.

Then, for any set of collocation parameters $\{c_i : \; 0 \le c_1 < \cdots < c_m \le 1\}$, the collocation error admits the estimate

$$\|y - u_h\|_\infty := \sup_{t\in(t_0,T]} |e_h(t)| \le Ch^m. \tag{3.32}$$

The constant C depends on the $\{c_i\}$ but not on $h := \max_{(n,\mu)} h_n^{(\mu)}$.

Although it follows from (3.28) and Theorem 3.3 that, in general, $\|\delta_h\|_\infty = \mathcal{O}(h^m)$ only, a judicious choice of the collocation parameters $\{c_i\}$ leads (not too surprisingly, if we look at the close connection between the degree of precision of interpolatory m-point quadrature formulas based on these abscissas and the variation-of-constants formula of Theorem 2.2 adapted to the error equation!) to *global superconvergence* on I for the *iterated collocation solution* u_h^{it}.

Theorem 3.4. Suppose that the assumptions (a)–(c) of Theorem 3.3 hold, but with $m+1$ replacing m in (a) and (b). If the collocation parameters $\{c_i\}$ are chosen so that the orthogonality condition

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)\, \mathrm{d}s = 0 \tag{3.33}$$

is satisfied, then the iterated collocation solution corresponding to the collocation solution $u_h \in S^{(-1)}_{m-1}(I_h)$ for (3.8) is globally superconvergent on I_h:

$$\|y - u_h^{\text{it}}\|_\infty \le C h^{m+1},$$

with C depending on the $\{c_i\}$ but not on h.

Proof. The key to the proof of Theorem 3.4 (and Theorem 3.6 below) on global superconvergence is the variation-of-constants formula (or 'resolvent representation') for e_h, together with the general global convergence result of Theorem 3.3 and the observation that

$$e_h^{\text{it}}(t) := y(t) - u_h^{\text{it}}(t) = e_h(t) - \delta_h(t), \quad t \in I.$$

For $t = t_n^{(\mu)} + v h_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ Theorem 2.2 yields, with e_h and δ_h replacing y, g and $g_0 = g$, respectively,

$$e_h^{\text{it}}(t) = \int_{\xi_\mu}^{t} R_1(t,s)\delta_h(s)\,\mathrm{d}s + \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\nu+1}} R_{\mu,\nu}(t,s)\delta_h(s)\,\mathrm{d}s + \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)\delta_h(s)\,\mathrm{d}s. \tag{3.34}$$

The integrals, having as lower and upper limits points of the (θ-invariant) mesh I_h, can be written as sums of integrals over individual micro-intervals $\bar{\sigma}_n$, and each of these integrals can then be replaced by the sum of an interpolatory m-point quadrature formula with respect to the collocation points in that interval and the corresponding quadrature error. The expression given by the quadrature formula has value zero, since $\delta_h(t) = 0$ for $t \in X_h$. Owing to the assumed regularity of the data (which is inherited on D by the resolvent R_1 and, piecewise on D, by the functions $R_{\mu,\nu}$, $Q_{\mu,\nu}$), the orthogonality condition (3.33) implies that all quadrature errors are $\mathcal{O}(h^{m+1})$. Here, we have used the result that, by definition, the defect δ_h and its derivatives $\delta_h^{(\nu)}$ $(\nu \le m+1)$, are uniformly bounded on each interval $I^{(\mu)}$.

It remains to deal with the integrals

$$\int_{t_n^{(\mu)}}^{t} R_1(t,s)\delta_h(s)\,\mathrm{d}s \quad \text{and} \quad \int_{t_n^{(\nu)}}^{\theta^{\mu-\nu}(t)} Q_{\mu,\nu}(t,s)\delta_h(s)\,\mathrm{d}s$$

(recall from (3.5) that $\theta^{\mu-\nu}(t) \in \sigma_n^{(\nu)}$ if $t \in \sigma_n^{(\mu)}$). As we have observed before, the defect δ_h induced by the collocation solution satisfies $\|\delta_h\|_\infty = \mathcal{O}(h^m)$. Thus, in the estimation of the above integrals (via the usual scaling) the uniform estimate for δ_h is multiplied by h, leading to the required $\mathcal{O}(h^{m+1})$-term in Theorem 3.4. □

Corollary 3.5. In the particular delay integral equation (3.22) assume that $g \in C^{m+1}(I)$ and $K \in C^{m+1}(\bar{D}_\theta)$, with $\bar{D}_\theta := \{(t,s) : \theta(t) \le s \le t,\ t \in I\}$. Then the iterated collocation solution based on $u_h \in S_{m-1}^{(-1)}(I_h)$ and defined by (3.27) has the global superconvergence property

$$\|y - u_h^{\mathrm{it}}\|_\infty \le Ch^{m+1}$$

provided the mesh I_h is θ-invariant, the $\{c_i\}$ underlying the set X_h of collocation points satisfy $J_0 = 0$ (*cf.* (3.33)), and $\phi \in C^{m+1}[\theta(t_0), t_0]$.

3.3.3. Local superconvergence results

The proof of the global superconvergence result in Theorem 3.4 indicates that we can readily refine it so as to establish stronger *local* superconvergence properties for u_h and u_h^{it} at the *mesh points* $t = t_n^{(\mu)}$.

Theorem 3.6. Let the given functions g, K_1, K_2 and ϕ in the delay integral equation (3.8) have continuous derivatives of order $m + \kappa$ in their respective domains I, D, D_θ and $[\theta(t_0), t_0]$, and assume that the delay function θ is subject to the conditions (D1)–(D3) of Section 3.2.2, with $d \ge m+\kappa$ in (D1). If $u_h \in S_{m-1}^{(-1)}(I_h)$ denotes the collocation solution for a θ-invariant mesh I_h, with corresponding iterated collocation solution u_h^{it}, and if the collocation parameters satisfy the orthogonality conditions

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^m (s - c_i)\,\mathrm{d}s = 0, \quad 0 \le \nu \le \kappa - 1,$$

with $J_\kappa \ne 0$, then

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{\mathrm{it}}| \le Ch^{m+\kappa}, \quad \text{for } h \in (0, \hat{h}).$$

If, in addition, we have $c_m = 1$ (implying $\kappa < m$), then u_h itself exhibits local superconvergence at the mesh points, that is,

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \le Ch^{m+\kappa}.$$

Proof. Our starting point is (3.34) in the proof of Theorem 3.4 where we now set $t = t_n^{(\mu)}$. Hence,

$$\begin{aligned} e_h^{\mathrm{it}}(t_n^{(\mu)}) = &\int_{\xi_\mu}^{t_n^{(\mu)}} R_1(t_n^{(\mu)}, s)\delta_h(s)\,\mathrm{d}s + \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\xi_{\mu+1}} R_{\mu,\nu}(t_n^{(\mu)}, s)\delta_h(s)\,\mathrm{d}s \\ &+ \sum_{\nu=0}^{\mu-1} \int_{\xi_\nu}^{\theta^{\mu-\nu}(t_n^{(\mu)})} Q_{\mu,\nu}(t_n^{(\mu)}, s)\delta_h(s)\,\mathrm{d}s, \end{aligned}$$

where $0 \le n < N$, $0 \le \mu \le M$, with $\theta^{\mu-\nu}(t_n^{(\mu)} = t_n^{(\nu)}$ (*cf.* (3.5)). Hence, the familiar quadrature argument is applicable: since the defect δ_h vanishes on X_h (and possesses uniformly bounded derivatives of order $m = \kappa$ on each $I^{(\mu)}$), and since the orthogonality and regularity conditions imply that the quadrature errors induced by the interpolatory m-point quadrature formulas based on the $\{c_i\}$ are all of order $\mathcal{O}(h^{m+\kappa})$, with the number $M+1$ of macro-intervals $I^{(\mu)}$ being finite, the first assertion in Theorem 4.6 follows immediately.

The second assertion is based on the fact that when $c_m = 1$, each mesh point $t_n^{(\mu)}$ $(1 \leq n \leq N)$ is a collocation point and thus $u_h^{\text{it}}(t_n^{(\mu)}) = u_h(t_n^{(\mu)})$, since $\delta_h(t_n^{(\mu)}) = 0$. Note also that $e_h^{\text{it}}(t_0) = 0$ because $u_h^{\text{it}}(t_0) = y(t_0^+)$. □

Corollary 3.7. Assume $\kappa = m$ in Theorem 3.6. Then collocation in $S_{m-1}^{(-1)}(I_h)$ at the Gauss points leads to an iterated collocation solution with the property that

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{\text{it}}(t)| \leq Ch^{2m},$$

while

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq Ch^{m} \quad \text{only.}$$

Corollary 3.8. Suppose that $\kappa = m-1$ and $c_m = 1$. The optimal order of convergence of the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ corresponding to the Radau II points is then given by

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h(t)| \leq Ch^{2m-1}.$$

Recall that we have $u_h^{\text{it}}(t) = u_h(t)$ for $t \in I_h \setminus \{t_0\}$ whenever $c_m = 1$ (*i.e.*, when $t_n \in X_h,\ n = 1, \ldots, N$).

We illustrate these results by an example; it is also introduced, in view of Sections 4 and 5, to remind the reader that the nature of a given delay $\tau(t)$ (non-vanishing versus vanishing) is often governed by location of the initial point t_0 in $I = [t_0, T]$.

Example 3.1. (Non-vanishing proportional delay) On $I = [t_0, T]$ with $t_0 > 0$, the delay function $\theta(t) = qt(0 < q < 1)$ corresponds to a non-vanishing delay $\tau(t)$ since

$$\theta(t) = qt = t - (1-q)t =: t - \tau(t),$$

with $\tau(t) \geq (1-q)t_0 > 0$ for $t \in I$. Hence, the primary discontinuity points $\{\xi_\mu\}$ are given by

$$\xi_\mu = q^{-\mu} t_0, \quad \mu \geq 0.$$

We will assume, for ease of exposition and without loss of generality, that T is such that $\xi_{M+1} = T$ for some $M > 1$. Hence, we may write

$$\xi_\mu = q^{M+1-\mu}T, \quad \mu = 0, 1, \ldots, M+1.$$

Suppose that the mesh I_h is constrained, and let each local mesh $I_h^{(\mu)}$ be *uniform*:

$$I_h^{(\mu)} := \{t_n^{(\mu)} := \xi_\mu + nh^{(\mu)} : \; n = 0, 1, \ldots, N \; (h^{(\mu)} = q^{-(\mu+1)}(1-q)t_0/N)\}.$$

A mesh of this type is often called a *quasi-geometric mesh* (see also Liu (1995*a*), Bellen, Guglielmi and Torelli (1997), Bellen (2001), Bellen, Brunner, Maset and Torelli (2002), and Guglielmi and Zennaro (2003). The linearity of θ then implies that I_h is θ-invariant, and the same is true for the set X_h of collocation points.

This choice of the local meshes defining I_h implies that

$$h = h^{(M)} = \frac{1}{N}(\xi_{M+1} - \xi_M) = (1-q)\frac{T}{N},$$

and

$$h^{(\mu)} = \frac{1}{N}(\xi_{\mu+1} - \xi_\mu) = q^{M+1-\mu-1}(1-q)\frac{T}{N}, \quad \mu = 0, 1, \ldots, M.$$

The result of, *e.g.*, Theorem 3.6 then becomes

$$\max_{t \in I_h \setminus \{t_0\}} |y(t) - u_h^{\text{it}}(t)| \le C(q)N^{-(m+\kappa)}.$$

Note that this result also holds for the delay VIE (3.22) on intervals $I = [t_0, T]$ with $t_0 > 0$.

3.3.4. Nonlinear delay VIEs

Since the extension of the convergence analysis presented in the previous sections to the general nonlinear version of (3.8),

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in (t_0, T], \tag{3.35}$$

is rather straightforward, we will omit it and instead focus on a class of nonlinear delay VIEs occurring frequently in applications. These functional equations have the form

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in (t_0, T], \tag{3.36}$$

where the (nonlinear) Volterra operator W_θ is of *Hammerstein type*:

$$(\mathcal{W}_\theta y)(t) := \int_{\theta(t)}^{t} k(t-s)G(s, y(s))\,\mathrm{d}s. \tag{3.37}$$

There are two ways to generate collocation approximations to solutions of Volterra–Hammerstein integral equations of the second kind. In the 'direct' approach discussed above we approximate y by $u_h \in S_{m-1}^{(-1)}(I_h)$, followed by the iterated collocation solution u_h^{it} based on u_h. Alternatively, we can resort to what is called *implicitly linear collocation*. Setting $z(t) := (\mathcal{N}y)(t) := G(t, y(t))$, where $\mathcal{N}$ is the *Niemytzki operator* (or substitution operator), the nonlinear delay VIE (3.36) becomes an *implicitly linear delay VIE* for z,

$$z(t) = G\Big(t, g(t) + \int_{\theta(t)}^{t} k(t-s)z(s)\,\mathrm{d}s\Big), \quad t \in (t_0, T], \tag{3.38}$$

with initial condition $z(t) = G(t, \phi(t)),\ t \in [\theta(t_0), t_0]$. The solution of the original DVIE is then obtained via the recursion

$$y(t) = g(t) + (\mathcal{L}_\theta z)(t), \quad t \in (t_0, T], \tag{3.39}$$

where $\mathcal{L}_\theta$ denotes the linear delay Volterra operator

$$(\mathcal{L}_\theta y)(t) := \int_{\theta(t)}^{t} k(t-s)z(s)\,\mathrm{d}s.$$

We note in passing that the survey paper by Brezis and Browder (1975) and the monograph by Krasnosel'skii and Zabreiko (1984) contain many results relevant in the analysis of solvability of Hammerstein integral equations and operator equations (*e.g.*, (3.38)) involving the Niemytzki operator. Compare also Kumar and Sloan (1987) and Brunner (1992) for details and additional references.

The solution z of (3.38) will be approximated by $z_h \in S_{m-1}^{(-1)}(I_h)$, using the same collocation points X_h as in the direct approach: it is defined by the *implicit linear collocation equation*

$$z_h(t) = G\Big(t, g(t) + \int_{\theta(t)}^{t} k(t-s)z_h(s))\,\mathrm{d}s\Big), \quad t \in X_h, \tag{3.40}$$

with initial values $z_h(t) = G(t, \phi(t)),\ t \in [\theta(t_0), t_0]$. This leads to the approximation y_h for the solution y of the orginal DVIE,

$$y_h(t) := g(t) + (\mathcal{L}_\theta z_h)(t), \quad t \in [t_0, T]. \tag{3.41}$$

Setting

$$z_h(t_n^{(\mu)} + v h_n^{(\mu)}) = \sum_{j=1}^{m} L_j(v) Z_{n,j}^{(\mu)}, \quad v \in (0,1], \quad \text{with } Z_{n,i}^{(\mu)} := z_h(t_{n,j}^{(\mu)}, \tag{3.42}$$

the computational forms of these equations at $t = t_{n,i}^{(\mu)}$ and at $t = t_n^{(\mu)} + v h_n^{(\mu)}$, respectively, are

$$Z_{n,i}^{(\mu)} = G\Big(t_{n,i}^{(\mu)}, g(t_{n,i}^{(\mu)}) + \bar{\Psi}_n^{(\mu-1)}(t_{n,i}^{(\mu)}) + h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^{c_i} k((c_i - s)h_n^{(\mu)}) L_j(s)\,\mathrm{d}s\Big) Z_{n,j}^{(\mu)}\Big) \tag{3.43}$$

($i = 1, \ldots, m$), where for $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \bar{\sigma}_n^{(\mu)}$ we have

$$\bar{\Psi}_n^{(\mu-1)}(t) := h_n^{(\mu-1)} \int_{\tilde{v}}^{1} k(t - t_n^{(\mu-1)} - sh_n^{(\mu-1)}) z_h(t_n^{(\mu-1)} + sh_n^{(\mu-1)})\,\mathrm{d}s$$

$$[2pt] \qquad + \int_{t_{n+1}^{(\mu-1)}}^{\xi_\mu} k(t-s) z_h(s)\,\mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} k(t-s) z_h(s)\,\mathrm{d}s,$$

and

$$y_h(t) = g(t) + \bar{\Psi}_n^{(\mu-1)}(t) + h_n^{(\mu)} \sum_{j=1}^{m} \Big(\int_0^{v} k((v-s)h_n^{(\mu)}) L_j(s)\,\mathrm{d}s\Big) Z_{n,j}^{(\mu)}, \quad v \in [0,1]. \tag{3.44}$$

Recall that the number $\tilde{v} \in [0,1]$ is obtained from

$$\theta(t_n^{(\mu)} + vh_n^{(\mu)}) =: t_n^{(\mu-1)} + \tilde{v}h_n^{(\mu-1)}, \quad v \in [0,1],$$

with $\tilde{v} = v$ if the delay function θ is linear.

The merits of this indirect collocation approach are twofold. Since in many applications the convolution kernel $k(t-s)$ is given by an elementary function like $\exp(\gamma(t-s))$ or $(t-s)^{-\alpha}$ ($\alpha < 1$), the integrals in (3.43) and (3.44) can be found analytically. Perhaps more importantly, the 'decoupling' of the nonlinear G and z in (3.38) implies that during the iteration process for solving the nonlinear algebraic systems (3.43) we do not have to recompute the integrals in each iteration step, in contrast to direct collocation for (3.36).

3.4. Collocation for VIDEs with delay arguments

3.4.1. The collocation equations

The description and analysis of piecewise polynomial collocation solutions for delay integral equations have provided all the essential ideas required to deal with collocation solutions for the initial-value problem for delay VIDEs,

$$y'(t) = f(t, y(t), y(\theta(t))) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I := [t_0, T],$$
$$y(t) = \phi(t), \quad t \in [\theta(t_0), t_0], \tag{3.45}$$

with Volterra integral operators $\mathcal{V}$ and $\mathcal{V}_\theta$ as in (3.8). The lag function θ

will again be assumed to satisfy conditions (D1)–(D3) of Section 3.2.2. We will usually employ its linear counterpart, corresponding to

$$f(t,y,z) = a(t)y + b(t)z + g(t), \tag{3.46}$$

to illustrate the essential ideas of the analysis.

The natural collocation space is now $S_m^{(0)}(I_h)$, and hence the collocation equation defining u_h with respect of the θ-invariant mesh I_h is

$$u_h(t) = f(t, u_h(t), u_h(\theta(t))) + (\mathcal{V}u_h(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h, \tag{3.47}$$

with $u_h(t) := \phi(t)$ if $t \le t_0$. For $t \in \sigma_n^{(\mu)}$ we define the lag term approximations

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} K_1(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t,s)u_h(s)\,\mathrm{d}s, \tag{3.48}$$

and

$$(\mathcal{V}_\theta u_h)(t) = \Psi_n^{(\mu-1)}(t) + \int_{t_n^{(\mu-1)}}^{\theta(t)} K_2(t,s)u_h(s)\,\mathrm{d}s. \tag{3.49}$$

In analogy to (3.14) we have

$$\Psi_n^{(\mu-1)}(t) = \int_{t_0}^{\xi_{\mu-1}} K_2(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} K_2(t,s)u_h(s)\,\mathrm{d}s.$$

With the local Lagrange representation of u_h on $\bar{\sigma}_n^{(\mu)}$,

$$u_h(t_n^{(\mu)} + sh_n^{(\mu)}) = y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(v) Y_{n,j}^{(\mu)}, \tag{3.50}$$
$$v \in [0,1], \quad \text{with} \quad Y_{n,j}^{(\mu)} := u_h'(t_n^{(\mu)}),$$

the computational form of (3.36) becomes

$$\begin{aligned} Y_{n,i}^{(\mu)} &= f(t_{n,i}^{(\mu)}, y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} a_{i,j} Y_{n,j}^{(\mu)}, u_h(\theta(t_{n,i}^{(\mu)}))) \\ &\quad + h_n^{(\mu)} \int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + sh_n^{(\mu)}) \left(y_n^{(\mu)} + h_n^{(\mu)} \sum_{j=1}^{m} \beta_j(s) Y_{n,j}^{(\mu)} \right) \mathrm{d}s \\ &\quad + F_n^{(\mu)}(t_{n,i}^{(\mu)}) + (\mathcal{V}_\theta u_h)(t_{n,i}^{(\mu)}), \quad i = 1, \ldots, m \end{aligned} \tag{3.51}$$

(which is reminiscent of the natural interpolant for an m-stage continuous implicit Runge–Kutta method for a DDE on a constrained mesh; see Bellen and Zennaro (2003, Chapter 6)). Recall from Lemma 3.1 that

$$\theta(t_{n,i}^{(\mu)}) = t_n^{(\mu-1)} + \tilde{c}_i h_n^{(\mu-1)},$$

which coincides with the collocation point $t_{n,i}^{(\mu-1)}$ $(i = 1, \dots, m)$ only if θ is linear.

On the subinterval $\bar{\sigma}_n^{(\mu)}$ the collocation solution $u_h \in S_m^{(0)}(I_h)$ is defined by the local representation (3.50). Hence, the solution $\mathbf{Y}_n^{(\mu)} \in \mathbb{R}^m$ of the linear algebraic system

$$\begin{aligned}[I_m - h_n^{(\mu)}(A_n^{(\mu)} + h_n^{(\mu)} C_n^{(\mu)})]\mathbf{Y}_n^{(\mu)} &= \mathbf{g}_n^{(\mu)} + \mathbf{G}_n^{(\mu)} + \boldsymbol{\kappa}_n^{(\mu)} y_n^{(\mu)} \\ &+ \mathbf{Q}_n^{(\mu-1)} + \tilde{\boldsymbol{\kappa}}_n^{(\mu-1)} y_n^{(\mu-1)} + (h_n^{(\mu-1)})^2 \tilde{C}_n^{(\mu-1)} \mathbf{Y}_n^{(\mu-1)},\end{aligned} \tag{3.52}$$

for $n = 0, 1, \dots, N-1$, $\mu = 0, 1, \dots, M$. The matrices in $L(\mathbb{R}^m)$ defining the left-hand side of (3.52) are

$$\begin{aligned}
A_n^{(\mu)} &:= \operatorname{diag}(a(t_{n,i}^{(\mu)}))A, \quad \text{with } \ A := (a_{i,j}); \\
\tilde{A}_n^{(\mu)} &:= \operatorname{diag}(b(t_{n,i}^{(\mu)}))\tilde{A}, \quad \text{with } \ \tilde{A} := (\beta_j(\tilde{c}_i); \\
C_n^{(\mu)} &:= \left(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) \beta_j(s) \, \mathrm{d}s \right)_{i,j=1}^{m} \\
\tilde{C}_n^{(\mu-1)} &:= \left(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) \beta_j(s) \, \mathrm{d}s \right)_{i,j=1}^{m},
\end{aligned}$$

and we have set

$$\begin{aligned}
\boldsymbol{\kappa}_n^{(\mu)} &:= \mathbf{a}_n^{(\mu)} + h_n^{(\mu)} \left(\int_0^{c_i} K_1(t_{n,i}^{(\mu)}, t_n^{(\mu)} + s h_n^{(\mu)}) \, \mathrm{d}s \right)_{i=1}^{m}, \\
\tilde{\boldsymbol{\kappa}}_n^{(\mu-1)} &:= \mathbf{b}_n^{(\mu)} + h_n^{(\mu-1)} \left(\int_0^{\tilde{c}_i} K_2(t_{n,i}^{(\mu)}, t_n^{(\mu-1)} + s h_n^{(\mu-1)}) \, \mathrm{d}s \right)_{i=1}^{m},
\end{aligned}$$

with

$$\mathbf{a}_n^{(\mu)} := \big(a(t_{n,i}^{(\mu)})\big)_{i=1}^{m}, \qquad \mathbf{b}_n^{(\mu)} := \big(b(t_{n,i}^{(\mu)})\big)_{i=1}^{m}.$$

The vectors $\mathbf{G}_n^{(\mu)}$ and $\mathbf{Q}_n^{(\mu-1)}$ are defined by

$$\begin{aligned}
\mathbf{G}_n^{(\mu)} &:= \big(F_n^{(\mu)}(t_{n,1}^{(\mu)}), \dots, F_n^{(\mu)}(t_{n,m}^{(\mu)})\big)^T, \\
\mathbf{Q}_n^{(\mu-1)} &:= \big(\Psi_n^{(\mu-1)}(t_{n,1}^{(\mu)}), \dots, \Psi_n^{(\mu-1)}(t_{n,m}^{(\mu)})\big)^T;
\end{aligned}$$

for $t \in \bar{\sigma}_n^{(\mu)}$ their components are given by

$$F_n^{(\mu)}(t) := \int_{t_0}^{\xi_\mu} K_1(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_\mu}^{t_n^{(\mu)}} K_1(t,s)u_h(s)\,\mathrm{d}s,$$

$$\Psi_n^{(\mu-1)}(t) := \int_{t_0}^{\xi_{\mu-1}} K_2(t,s)u_h(s)\,\mathrm{d}s + \int_{\xi_{\mu-1}}^{t_n^{(\mu-1)}} K_2(t,s)u_h(s)\,\mathrm{d}s,$$

respectively (*cf.* (3.18) and (3.14)).

Theorem 3.9. Assume that the given functions a, b, g, K_1, K_2 describing the linear delay VIDE (3.45), (3.46) are continuous on their respective domains, and let the delay functions θ be subject to the hypotheses (D1)–(D3) in Section 3.2.2. Then there exists a $\bar{h} > 0$ so that for any θ-invariant mesh I_h with $h \in (0, \bar{h})$ and any initial function $\phi \in C[\theta(t_0), t_0]$ each of the linear algebraic systems (3.52) possesses a unique solution $Y_n^{(\mu)} \in \mathbb{R}^m$. Therefore, the collocation equation (3.47) defines a unique collocation solution $u_h \in S_m^{(0)}(I_h)$ whose local representation on $\bar{\sigma}_n^{(\mu)}$ is given by (3.50).

3.4.2. Global convergence results

The collocation error $e_h := y - u_h$ associated with the collocation solution $u_h \in S_m^{(0)}(I_h)$ to the linear DVIDE (3.45), (3.46) solves the initial-value problem

$$\begin{aligned} &e_h'(t) = a(t)e_h(t) + b(t)e_h(\theta(t)) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I, \\ &e_h(t) = 0, \quad t \in [\theta(t_0), t_0], \end{aligned} \tag{3.53}$$

where the defect δ_h vanishes on X_h, the set of collocation points. For $t \in I^{(\mu)} := [\xi_\mu, \xi_{\mu+1}]$ we write the above error equation in the form

$$e_h'(t) = a(t)e_h(t) + \delta_h(t) + G_\mu(t) + \int_{\xi_\mu}^{t} K_1(t,s)e_h(s)\,\mathrm{d}s, \quad t \in I^{(\mu)}, \tag{3.54}$$

with given initial value $e_h(\xi_\mu)$ and lag term

$$G_\mu(t) := b(t)e_h(\theta(t)) + \int_{t_0}^{\xi_\mu} K_1(t,s)e_h(s)\,\mathrm{d}s + (\mathcal{V}_\theta e_h)(t).$$

When $\mu = 0$ we have

$$e_h'(t) = a(t)e_h(t) + \delta_h(t) + \int_{t_0}^{t} K_1(t,s)e_h(s)\,\mathrm{d}s, \quad t \in I^{(0)}, \tag{3.55}$$

since the initial condition $e_h(t) = 0,\ t \le 0$ implies $G_0(t) = 0$ in $[\theta(t_0), t_0]$.

Thus, on the first macro-interval $I^{(0)}$ the gobal convergence result for classical VIDEs holds: under appropriate assumptions on the regularity of the solution (see Theorem 3.12 below) the collocation error can be estimated by

$$\|e_h^{(\nu)}\|_{0,\infty} := \sup_{t\in I^{(0)}} |e_h^{(\nu)}(t)| \le C_\nu (h^{(0)})^m, \quad \nu = 0, 1.$$

It follows in particular that $e_h^{(\nu)}(\xi_1) = \mathcal{O}((h^{(0)})^m)$.

This result allows us to derive an similar global error estimate on each macro-interval $I^{(\mu)}$ $(1 \le \mu \le M)$. We leave the detailed steps in this recursive argument to the reader and simply summarize the result in the following theorem.

Theorem 3.10. Let the following conditions hold.

(a) a, b, $g \in C^m(I)$, and $\phi \in C^{m+1}[\theta(t_0), t_0]$.

(b) $K_1 \in C^m(D)$, $K_2 \in C^m(D_\theta)$.

(c) θ satisfies conditions (D1)–(D3) of Section 3.2.2, with $d \ge m$ in (D1).

(d) $u_h \in S_m^{(0)}(I_h)$ is the collocation solution to the delay VIDE (3.45), (3.46) where I_h is θ-invariant and $h \in (0, \bar{h})$ so that the linear algebraic systems (3.52) all have unique solutions.

Then the estimates

$$\|y^{(\nu)} - u_h^{(\nu)}\|_\infty \le C_\nu h^m, \quad \nu = 0, 1 \tag{3.56}$$

hold for any set $\{c_i\}$ of distinct collocation parameters in $[0, 1]$. The constants C_ν depend on these parameters but are independent of h.

As for second-kind VIEs with non-vanishing delays, a gain of one can be achieved in the global order of convergence of u_h by a judicious choice of the $\{c_i\}$, thus extending the global superconvergence result for classical VIDEs (Brunner and van der Houwen (1986) or Brunner (2004*b*, Chapter 3)).

Theorem 3.11. Let the assumed degree of regularity for the given functions in the initial-value problem for the linear delay VIDE (3.45), (3.46) be raised by one (to $m+1$ and $m+2$, respectively) in Theorem 3.10. If the collocation parameters satisfy the orthogonality condition

$$J_0 := \int_0^1 \prod_{i=1}^m (s - c_i)\,\mathrm{d}s = 0$$

then, for all θ-invariant meshes I_h with $h \in (0, \bar{h})$, the collocation solution $u_h \in S_m^{(0)}(I_h)$ is globally superconvergent on I, that is,

$$\|y - u_h\|_\infty \le C h^{m+1}, \tag{3.57}$$

with C depending on the $\{c_i\}$ but not on h.

Proof. The key to establishing this global superconvergence result (and the local superconvergence results in the next section) is the variation-of-constants result of Theorem 2.7, where y and g are replaced, respectively, by e_h and δ_h, and where the initial conditions are given by $e_h(t) = 0$ $(t \le t_0)$ and $e_h(\xi_\mu) = \mathcal{O}((h^{(\mu)})^{m+1})$ $(1 \le \mu \le M;\ h^{(\mu)} \le h)$. We note once more that the image of a point $t = t_n^{(\mu)} + vh_n^{(\mu)} \in \sigma_n^{(\mu)}$ under $\theta^{\mu-\nu}$ $(0 \le \nu \le \mu - 1)$ is given either by $t_n^{(\nu)} + vh_n^{(\nu)}$ $(v \in [0,1])$ if θ is linear, or by $t_n^{(\nu)} + \tilde{v}h_n^{(\nu)}$ (for some $\tilde{v} \in [0,1]$, with $\tilde{v} \ne v$) if θ is *nonlinear.*

Details of the proof are left as an exercise. □

Remark. The convergence results of Theorems 3.12 and 3.13 contain, as special cases, global convergence and superconvergence results for DDEs (for $K_i = 0$ on D and D_θ, respectively).

3.4.3. Local superconvergence results

In the previous section we described the foundation for proving optimal superconvergence results on I_h for the collocation solution $u_h \in S_m(^{(0)}(I_h)$ to the linear delay VIDE (3.45), (3.46): it is given by the variation-of-constants formula (or 'resolvent representation') for the collocation error $e_h := y - u_h$. The essential ingredients of the proof of the local superconvergence result are thus all in place: the θ-invariance of the mesh I_h and the resulting mapping (3.4), (3.5) of mesh points $t_n^{(\mu)}$ into corresponding previous mesh points $t_n^{(\nu)}$ (which is of course true regardless of whether the delay function θ is linear or nonlinear) and the order of the quadrature errors corresponding to the interpolatory m-point quadrature formulas based on the collocation points and depending on the familiar orthogonality conditions for the collocation parameters $\{c_i\}$. Thus, without any more ado we state the following result.

Theorem 3.12. Let the following be satisfied.

(a) The given functions a, b, g and K_1, K_2 in the DVIDE (3.45), (3.46) are in $C^{m+\kappa}$ on their respective domains, for some κ with $1 \le \kappa \le m$, as specified in (d).

(b) The delay function θ is subject to (D1)–(D3), with $d \ge m+\kappa$ in (D1).

(c) $u_h \in S_m^{(0)}(I_h)$ is the collocation solution, with θ-invariant mesh I_h, for the given delay VIDE.

(d) The collocation parameters $\{c_i\}$ are such that the orthogonality conditions of Theorem 3.6,

$$J_\nu := \int_0^1 s^\nu \prod_{i=1}^{m} (s - c_i)\, \mathrm{d}s = 0, \quad \nu = 0, 1, \ldots, \kappa - 1,$$

with $J_\kappa \ne 0$, hold.

Then, for all $h \in (0, \bar{h})$ (Theorem 3.9), the collocation error $e_h := y - u_h$ satisfies

$$\max_{t \in I_h} |e_h(t)| \leq C h^{m+\kappa}, \tag{3.58}$$

for some constant C which depends on the $\{c_i\}$ but not on h.

If, in addition, $c_m = 1$ (implying $\kappa \leq m-1$), then we also have

$$\max_{t \in I_h \setminus \{t_0\}} |e_h'(t)| \leq C_1 h^{m+\kappa}. \tag{3.59}$$

4. Basic theory of Volterra functional integral equations II: (vanishing) proportional delays

4.1. The pantograph equation: ca. *1971*

The linear DDE with constant coefficients,

$$y'(t) = ay(t) + by(qt), \quad t \in I := [0, T], \quad 0 < q < 1, \tag{4.1}$$

arose in the mathematical modelling of the wave motion in the supply line to an overhead current collector (*pantograph*) of an electric locomotive (see Ockendon and Tayler (1971) and Fox, Mayers, Ockendon and Tayler (1971); also Tayler (1986, pp. 40–45, 50–53)). The resulting *pantograph equation* is a (seemingly!) very simple example of a DDE with *vanishing* variable delay: here, we have $\theta(t) = t - \tau(t)$, with $\tau(t) = (1-q)t \geq 0$.

A special case of (4.1) is the 'pure delay' equation

$$y'(t) = by(qt), \quad t \geq 0, \quad y(0) = y_0, \quad b \neq 0. \tag{4.2}$$

Its (unique) solution is given by

$$y(t) = \sum_{j=0}^{\infty} \frac{q^{j(j-1)/2}}{j!} (bt)^j \cdot y_0, \quad t \geq 0. \tag{4.3}$$

The following result can be found in Kato and McLeod (1971); compare also Frederickson (1971), Morris, Feldstein and Bowen (1972), Carr and Dyson (1976), Derfel (1990), Iserles (1993), and Terjéki (1995).

Theorem 4.1. For any $q \in (0,1)$ and any y_0 the delay differential equation (4.1) possesses a unique solution $y \in C^1(I)$ with $y(0) = y_0$, regardless of the choice of a, $b \neq 0$, and $T > 0$. It is given by

$$y(t) = \sum_{n=0}^{\infty} \gamma_n(q) t^n,$$

where

$$\gamma_n(q) := \frac{1}{n!} \prod_{j=1}^{n} (a + bq^{j-1}).$$

Proof. We apply Picard iteration to the equivalent Volterra integral equation,

$$y(t) = y_0 + \int_0^t (ay(s) + by(qs))\,\mathrm{d}s, \quad t \in I.$$

It can be shown that the resulting sequence $\{y_n(t)\}$ ($n \geq 0$, $y_0(t) := y_0$) converges uniformly on any interval I. Moreover, setting

$$y(t) := \sum_{n=0}^{\infty} \gamma_n(q) t^n,$$

one verifies that the power series has infinite radius of convergence, since its coefficients satisfy

$$\frac{\gamma_n}{\gamma_{n-1}} = \frac{1}{n}[a + bq^{n-1}], \quad n \geq 1. \qquad \square$$

Remarks. **(1)** The survey paper by Iserles (1993) presents an illuminating introduction into the complex world of solutions to (4.1) and its generalizations; it also contains an extensive bibliography. The pantograph equation and its matrix version is also studied, within the framework of Volterra functional integral equations, in Chambers (1990, pp. 40–43).

(2) The above result on the existence and uniqueness of solutions remains true for (4.1) with variable coefficients a, $b \in C(I)$. More precisely, if a, $b \in C^m(I)$ then, for any $q \in (0, 1)$ and any y_0, the solution y lies in $C^{m+1}(I)$. Properties of solutions of nonlinear versions of these equations (*e.g.*, Riccati-type equations) can be found in Iserles (1994*a*) and Iserles and Terjéki (1995).

(3) These results confirm a crucial difference between the regularity of solutions to DDEs with non-vanishing delays and those of pantograph-type DDEs: for the latter, smooth data lead to solutions that are smooth on the *entire* interval $[0, T]$. In particular, solutions to (4.2) are *entire functions* of *order zero*. It follows from classical complex function theory (Ahlfors' theorem) that an entire function of order zero cannot have finite asymptotes. This implies that, for $b < 0$, nontrivial solutions of (4.2) are not bounded on $\mathbb{R}^+$; also, the number of sign changes (zeros) is infinite. (See also Iserles (1993), Iserles (1997*b*), and Liu (1997).) To give the reader an idea of how these zeros depend on q, Table 4.1 exhibits a representative sample of zeros of y. Additional information (for $q = 1/4$, $q = 3/4$) can be found in Iserles (1993, p. 5).

The maximum of $|y(t)|$ in the interval given by the last listed zero and the following one exceed 10^{15}. We note in passing that the papers by Iserles (1997*b*) and Liu (1997) nicely describe and illustrate the various difficulties

Table 4.1. Zeros of $y(t)$ for $b = -1$.

$q = 0.05$	$q = 0.5$	$q = 0.95$
$z_1 = 1.02631$	$z_1 = 1.48808$	$z_1 = 8.96684$
$z_2 = 40.3651$	$z_2 = 4.88114$	$z_2 = 10.8942$
$\vdots$	$\vdots$	$\vdots$
$z_3 = 1205.57$	$z_{10} = 5223.38$	$z_{46} = 5258.99$

one encounters in the long-time approximation of solutions to the 'innocent' pantograph equation (4.1).

The reader interested in details on the asymptotic distribution of the zeros of such solutions may wish to consult the 1992 paper by Elbert (which includes a reference to the first study of this subject, a 1967 report by Feldstein and Kolb). The papers by Iserles (1994*b*), Iserles and Terjéki (1995), and Feldstein and Liu (1998) contain a wealth of results on *nonlinear* pantograph DDEs, including Riccati-type functional equations.

4.2. Linear Volterra integral equations with proportional delays

We now return to one of the particular delay VIEs considered by Andreoli (1913, 1914), and to his important remark regarding the effect the (vanishing) proportional delay has on the representation of its solution. In the notation employed in this paper this equation is

$$y(t) = g(t) + \int_0^{qt} K(t,s)y(s)\,\mathrm{d}s, \quad t \in I := [0,T], \quad 0 < q < 1, \qquad (4.4)$$

where g and K are continuous functions.

Theorem 4.2. Let g and K in (4.4) satisfy $g \in C(I)$ and $K \in C(D_\theta)$, where $D_\theta := \{(t,s) : \ 0 \le s \le \theta(t), \ t \in I\}$. Then, for any $\theta(t) := qt$ with $q \in (0,1)$, the delay integral equation (4.4) possesses a unique solution $y \in C(I)$. This solution is given by

$$\begin{aligned} y(t) &= g(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} K_n(t,s)g(s)\,\mathrm{d}s \\ &= g(t) + \int_0^t \left(\sum_{n=1}^{\infty} q^n K_n(t,q^n s)g(q^n s) \right) \mathrm{d}s, \quad t \in I. \end{aligned} \qquad (4.5)$$

The iterated kernels $K_n(t,s) = K_n(t,s;q)$ $(n \geq 1)$ are obtained recursively by

$$K_{n+1}(t,s) := \int_{q^{-n}s}^{qt} K(t,v)K_n(v,s)\,\mathrm{d}v, \quad (t,s) \in D_\theta^{(n+1)}, \quad n \geq 1,$$

with $K_1(t,s) := K(t,s)$ and

$$D_\theta^{(k)} := \{(t,s) : \ 0 \leq s \leq q^k t, \ t \in I\}.$$

Remark. For $q = 1$, the solution representation (4.5) reduces to the classical 'separable' expression involving the resolvent kernel $R(t,s)$ (as the limit of the Neumann series) of $K(t,s)$. Theorem 4.2 shows that for $0 < q < 1$ such a resolvent representation of the solution no longer exists: the values of the iterated kernels 'overlap' with those of g. However, the infinite series in (4.5) still converges uniformly on any compact interval I, as Lemma 4.3 below will make clear.

Proof. The Picard iteration process we applied to the integrated form of the pantograph DDE can of course be used for the delay VIE (4.4), with suitably adapted Dirichlet's formula when changing the order of integration in the double integrals: here, the resulting limits of integration now depend on the iteration number n. To see this in some more detail, we have, setting $y_0(t) := g(t)$,

$$y_1(t) := g(t) + \int_0^{qt} K_1(t,s)g(s)\,\mathrm{d}s$$

and hence

$$\begin{aligned} y_2(t) &:= g(t) + \int_0^{qt} K_1(t,s)\Big(g(s) + \int_0^{qs} K_1(s,v)g(v)\,\mathrm{d}v\Big)\,\mathrm{d}s \\ &= g(t) + \int_0^{qt} K_1(t,s)g(s)\,\mathrm{d}s + \int_0^{qt}\Big(\int_{q^{-1}v}^{qt} K_1(t,s)K_1(s,v)\,\mathrm{d}s\Big)y(v)\,\mathrm{d}v. \end{aligned}$$

It is now easily verified by induction that the iterated kernels $K_n(t,s)$ of the given kernel $K(t,s) =: K_1(t,s)$ are generated recursively by

$$K_{n+1}(t,s) = \int_{q^{-n}s}^{qt} K(t,v)K_n(v,s)\,\mathrm{d}v, \quad (t,s) \in D_\theta^{(n+1)}, \quad n \geq 1$$

(see also Chambers (1990)). Hence the iterate $y_k(t)$ can be expressed in the form (4.5) where the index of summation ranges from 1 to k.

Lemma 4.3. Uniform bounds on $I = [0,T]$ for the iterated kernels $K_n(t,s)$ defined in Theorem 4.2 are given by

$$|K_n(t,s)| \leq \frac{q^{n(n-1)/2}}{(n-1)!}T^{n-1}\bar{K}_\theta^n, \quad (t,s) \in D_\theta^{(n)}, \quad n \geq 1,$$

where we have set $\bar{K}_\theta := \max_{(D_\theta)} |K(t,s)|$.

We leave the proof of this simple result, as well as that of the uniqueness of the solution (4.5), as an exercise. □

The existence, uniqueness and regularity properties hold also for the more general linear delay VIE with proportional delay,

$$y(t) = g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{4.6}$$

corresponding to the Volterra integral operators

$$(\mathcal{V}y)(t) := \int_0^t K_1(t,s)y(s)\,\mathrm{d}s, \qquad (\mathcal{V}_\theta y)(t) := \int_0^{\theta(t)} K_2(t,s)y(s)\,\mathrm{d}s,$$

with $\theta(t) := qt$ $(0 < q < 1)$, $K_1 \in C(D)$ and $K_2 \in C(D_\theta)$.

Theorem 4.4. Assume that $K_1 \in C^d(D)$ and $K_2 \in C^d(D_\theta)$, for some $d \geq 0$. Then the delay integral equation (4.6) with $\theta(t) = qt$ $(0 < q < 1)$ has a unique solution $y \in C^d(I)$ for any g with $g \in C^d(I)$.

Proof. Theorem 4.2 shows that the iterated kernels $K_n(t,s)$ associated with the kernel K of the special delay integral equation (4.4) inherit the regularity of K. Since the additional term $(\mathcal{V}y)(t)$ in the general linear delay VIE (4.6) will not lead to lower regularity in the Picard iteration process, the assertion of Theorem 4.4 follows from the uniform convergence of the Picard iterates on I, for any $q \in (0,1)$. □

We shall see in Section 4.5 that this regularity result can also be derived by means of embedding techniques.

Remark. The paper by Morris *et al.* (1972, pp. 518–523) contains an illuminating discussion of the connection between general pantograph DDEs and certain Volterra integral and integro-differential equations with (multiple) proportional delays. Compare also Iserles and Liu (1994).

4.3. First-kind Volterra integral equations with vanishing delays

In Section 1.1.1 we briefly alluded to the fact that in his *Nota I* of 1896 Volterra studied the problem of 'inverting' definite integrals of the form

$$(\mathcal{V}y)(t) := \int_0^t K(t,s)y(s)\,\mathrm{d}s = g(t), \quad t \in I := [0,T], \quad g(0) = 0,$$

where $K \in C(D)$, and that he then went on (Volterra (1897), also Volterra (1913, pp. 92–101)) to analyse the more general functional equation

$$(\mathcal{W}_\theta y)(t) = g(t), \quad t \in I, \quad g(0) = 0, \tag{4.7}$$

where the Volterra integral operator $\mathcal{W}_\theta : \ C(I) \to C(I)$ is

$$(\mathcal{W}_\theta \phi)(t) := \int_{\theta(t)}^{t} K(t,s)\phi(s)\,\mathrm{d}s, \quad \text{with} \;\; \theta(t) := qt \; (0 < q < 1). \qquad (4.8)$$

Under suitable conditions on K and g this equation can be transformed into the equivalent second-kind equation

$$K(t,t)y(t) - qK(t,qt)y(qt) + \int_{qt}^{t} \frac{\partial K(t,s)}{\partial t} y(s)\,\mathrm{d}s = g'(t), \quad t \in I \qquad (4.9)$$

(see also Fenyö and Stolle (1984, pp. 324–327) and Brunner (1997*b*)), to which Picard iteration techniques can be applied. This reformulation was the basis for Volterra's 1897 result, which we now state. We set $\bar{D}_\theta := \{(t,s) : \ 0 \le \theta(t) \le s \le t \le T\}$.

Theorem 4.5. Assume:

(a) $g \in C^1(I)$, with $g(0) = 0$;
(b) $K \in C^{(}\bar{D}_\theta)$, $\partial K/\partial t \in C(\bar{D}_\theta)$, with $|K(t,t)| \ge k_0 > 0 \; (t \in I)$.

Then, for each $\theta(t) = qt$ with $q \in (0,1)$, the first-kind delay integral equation (4.7) possesses a unique solution $y \in C(I)$.

The above result was generalized by Lalesco (1908, 1911) and – much later – by Denisov and Korovin (1992) and Denisov and Lorenzi (1995). From the latter paper we cite the following result.

Theorem 4.6. Assume the lag function θ satisfies

(a) $\theta \in C^3(I)$, with $\theta(0) = 0$, $\theta'(0) = 1$, $\theta''(0) < 0$, $\theta(t) < t \; (t \in (0,T])$, $\theta'(t) > 0$ for $t \in I$,

and let

(b) $g \in C^2(I)$, with $g(0) = g'(0) = 0$;
(c) $K \in C^3(\bar{D}_\theta)$, with $|K(t,t)| \ge k_0 > 0 \; (t \in I)$.

Then the first-kind delay integral equation $(\mathcal{W}_\theta y)(t) = g(t)$ has a unique solution $y \in C(I)$.

Remark. A similar result was proved in Denisov and Korovin (1992), but under the hypothesis that $\theta'(0) < 1$. If, as in the above theorem, $\theta'(0) = 1$, the domain $\bar{D}_\theta$ has a cusp at the point $(t,s) = (0,0)$, and new techniques are needed to deal with this situation. We note that the case $\theta'(0) = 1$ was already treated, albeit in a somewhat sketchy way, by Lalesco in 1911.

4.4. Volterra integro-differential equations with proportional delays

In order to obtain some first insight into the properties of solutions of linear VIDEs with proportional delays we will first consider the 'pure delay' problem

$$y'(t) = g(t) + \int_0^{qt} K(t,s)y(s)\,\mathrm{d}s, \quad t \in I := [0,T], \quad y(0) = y_0, \tag{4.10}$$

assuming that $g \in C(I)$, $K \in C(D_\theta)$, with $\theta(t) = qt$ and $0 < q < 1$. This initial-value problem is equivalent to the delay VIE

$$y(t) = g_0(t) + \int_0^{qt} H(t,s;q)y(s)\,\mathrm{d}s, \quad t \in I, \tag{4.11}$$

where

$$g_0(t) := y_0 + \int_0^t g(s)\,\mathrm{d}s, \qquad H(t,s;q) := \int_{q^{-1}s}^t K(v,s)\,\mathrm{d}v.$$

We now apply Theorem 4.2: setting $H_1(t,s) := H(t,s;q)$, and denoting by $H_n(t,s)$ the corresponding iterated kernels, the (unique) solution y of (4.11) (which, since g_0 and $H(\cdot,\cdot;q)$ are continuously differentiable functions, lies in $C^1(I)$) can be expressed in the form

$$y(t) = g_0(t) + \sum_{n=1}^{\infty} \int_0^{q^n t} H_n(t,s)g_0(s)\,\mathrm{d}s, \quad t \in I,$$

where the infinite series converges absolutely and uniformly. If we now substitute the expressions for $g_0(t)$, an obvious rearrangement (using Dirichlet's formula) leads to the following result.

Theorem 4.7. Under the above assumptions on g and K, the unique solution $y \in C^1(I)$ to the initial-value problem (4.10) has the representation

$$y(t) = \left(1 + \sum_{n=1}^{\infty} \tilde{H}_n(t,0)\right) y_0 + \sum_{n=0}^{\infty} \int_0^{q^n t} \tilde{H}_n(t,s)g(s)\,\mathrm{d}s, \quad t \in I.$$

Here, we have set $\tilde{H}_0(t,s) := 1$ $((t,s) \in D)$,

$$\tilde{H}_n(t,s) := \int_s^{q^n t} H_n(t,v)\,\mathrm{d}v, \quad n \geq 1,$$

and we note that

$$\tilde{H}_n(t,0) = \int_0^{q^n t} H_n(t,v)\,\mathrm{d}v, \quad n \geq 1.$$

The initial-value problem for the *general* linear VIDE with proportional delay,

$$y'(t) = a(t)y(t) + b(t)y(qt) + g(t) + (\mathcal{V}y)(t) + (\mathcal{V}_\theta y)(t), \quad t \in I, \tag{4.12}$$

with $\theta(t) = qt$ $(0 < q < 1)$, is equivalent to the delay VIE

$$\begin{aligned} y(t) &= g_0(t) + \int_0^t \left(a(s) + \int_s^t K_1(v,s)\,\mathrm{d}v \right) y(s)\,\mathrm{d}s \\ &\quad + \int_0^{qt} \left((1/q)b(s/q) + \int_{q^{-1}s}^t K_2(v,s)\,\mathrm{d}v \right) y(s)\,\mathrm{d}s \\ &=: g_0(t) + \int_0^t G_1(t,s)y(s)\,\mathrm{d}s + \int_0^{qt} G_2(t,s;q)y(s)\,\mathrm{d}s, \end{aligned}$$

where

$$g_0(t) := y_0 + \int_0^t g(s)\,\mathrm{d}s.$$

The regularity of the kernels G_1 and $G_2(\cdot\,;\cdot\,;q)$ is determined by that of the original data a, b and K_1, K_2. Thus, Theorem 4.4 implies the following result.

Theorem 4.8. Assume:

(a) $a,\ b,\ g \in C^d(I)$ for some $d \geq 0$;
(b) $K_1 \in C^d(D)$ and $K_2 \in C^d(D_\theta)$.

Then, for each initial value y_0, the delay VIDE (4.12) possesses a unique solution $y \in C^{d+1}(I)$.

4.5. Embedding techniques

The embedding of a (proportional) delay differential equation into an infinite system of ordinary differential equations was studied in detail by Feldstein, Iserles and Levin (1995). The motivation behind their approach was to explore another way for obtaining results on the asymptotic stability (or the boundedness) of solutions of such DDEs, and for constructing feasible methods for their numerical solution. It also permits the derivation of existence and regularity results for the exact solutions.

Here, we extend these embedding techniques to the delay Volterra integral equation (4.6) and to the delay Volterra integro-differential equation (4.12). Note that these Volterra functional equations contain the important special cases characterized by $K_2(t,s) = -K_1(t,s) =: -K(t,s)$:

$$y(t) = g(t) + (\mathcal{W}_\theta y)(t), \quad t \in I, \tag{4.13}$$

and

$$y'(t) = a(t)y(t) + b(t)y(qt) + (\mathcal{W}_\theta y)(t), \quad t \in I, \quad y(0) = y_0, \tag{4.14}$$

corresponding to the delay Volterra operator $\mathcal{W}_\theta$ defined in (4.8). The following embedding results (which can be extended to the nonlinear counterparts of the above pantograph-type Volterra equations) contain the key not only to establishing results on the existence, uniqueness, and regularity of solutions but also to the analysis of the local superconvergence properties of collocation solutions to such functional equations.

4.5.1. Embedding results for the delay VIE (4.6)

Lemma 4.9. The delay VIE (4.6) can be embedded into an infinite-dimensional system of 'classical' VIEs of the second kind,

$$z_\nu(t) = g_\nu(t) + \int_0^t (K_{1,\nu}(t,s) z_\nu(s) + K_{2,\nu}(t,s) z_{\nu+1}(s))\,\mathrm{d}s, \quad \nu \in \mathbb{N}_0, \tag{4.15}$$

where

$$z_\nu(t) := y(q^\nu t), \qquad g_\nu(t) := g(q^\nu t)$$

and

$$K_{1,\nu}(t,s) := q^\nu K_1(q^\nu t, q^\nu s), \qquad K_{2,\nu}(t,s) := q^{\nu+1} K_2(q^\nu t, q^{\nu+1} s).$$

The proof of this embedding result is straightforward and will thus be left as an exercise.

Consider now the *truncated* (finite) system corresponding to (4.15),

$$z_{M,\nu}(t) = g_\nu(t) + \int_0^t (K_{1,\nu}(t,s) z_{M,\nu}(s) + K_{2,\nu}(t,s) z_{M,\nu+1}(s))\,\mathrm{d}s, \tag{4.16}$$
$$\nu = 0, 1, \ldots, M-1,$$

$$z_{M,M}(t) = g_M(t) + \int_0^t K_{1,M}(t,s) z_{M,M}(s)\,\mathrm{d}s, \quad t \in I. \tag{4.17}$$

Lemma 4.10. Assume that $g \in C(I)$, $K_1 \in C(D)$, $K_2 \in C(D_\theta)$. Then, for $\nu = M, M-1, \ldots, 0$, the (unique) solution of (4.16) and (4.17) satisfies

$$\|z_\nu - z_{M,\nu}\|_\infty \le C q^{\tilde{M}}, \quad \text{with} \quad \tilde{M} \ge M+1.$$

Proof. Setting $\varepsilon_{M,\nu} := z_\nu - z_{M,\nu}$, it follows from (4.16) and (4.17) that

$$\varepsilon_{M,\nu}(t) = \int_0^t K_{1,\nu}(t,s) \varepsilon_{M,\nu}(s)\,\mathrm{d}s + \Phi_{M,\nu}(t), \quad t \in I, \tag{4.18}$$

for $\nu = 0, 1, \ldots, M$, with

$$\Phi_{m,\nu}(t) := \begin{cases} \displaystyle\int_0^t K_{2,M}(t,s) z_{M+1}(s)\,\mathrm{d}s, & \text{if } \nu = M, \\ \displaystyle\int_0^t K_{2,\nu}(t,s) \varepsilon_{M,\nu+1}(s)\,\mathrm{d}s, & \text{if } M-1 \ge \nu \ge 0. \end{cases}$$

Let $R_{1,\nu} = R_{1,\nu}(t,s)$ denote the resolvent kernel associated with the kernel $K_{1,\nu}$ in (4.15); we know from classical Volterra theory that $K_1 \in C(D)$

implies $R_{1,\nu} \in C(D)$ for all $\nu \geq 0$. The (unique) solution of the finite system (4.18) may thus be written as

$$\varepsilon_{M,\nu}(t) = \int_0^t R_{1,\nu}(t,s)\Phi_{M,\nu}(s)\,\mathrm{d}s + \Phi_{M,\nu}(t), \quad t \in I, \tag{4.19}$$

for $\nu = M,\ M-1, \ldots, 0$. Since $|K_{2,\nu}(t,s)| \leq \bar{K}_2 q^{\nu+1}$, $(t,s) \in D_\theta$, where $\bar{K}_2 := \max_{D_\theta} |K_2(t,s)|$, setting $\nu = M$ in (4.18) leads to

$$|\varepsilon_{M,M}(t)| \leq C q^{M+1}, \quad t \in I.$$

Thus, assuming that $\|\varepsilon\|_\infty \leq C q^{\tilde{M}}$ ($\tilde{M} \geq M$) for $\nu = M,\ M-1, \ldots, M_0+1$, we find

$$|\Phi_{M_0,\nu}(t)| \leq \bar{K}_2 T q^{\nu+1} C_0 q^{\tilde{M}} =: C q^{\tilde{M}+\nu+1}, \quad \nu \geq \tilde{M}+1,$$

and hence,

$$|\varepsilon_{M,M_0}(t)| \leq C q^{\tilde{M}}, \quad t \in I, \quad \text{with} \ \ \tilde{M} \geq M+1.$$

This establishes the uniform bounds in Lemma 4.10. □

4.5.2. Embedding results for the delay VIDE (4.12)

Lemma 4.11. The delay VIDE (4.12) can be embedded into an infinite-dimensional system of 'classical' VIDEs, namely,

$$z'_\nu(t) = \tilde{a}_\nu(t) z_\nu(t) + \tilde{b}_\nu(t) z_{\nu+1}(t) + \int_0^t \big(\tilde{K}_{1,\nu}(t,s) z_\nu(s) + \tilde{K}_{2,\nu}(t,s) z_{\nu+1}(s)\big)\,\mathrm{d}s, \tag{4.20}$$

for $\nu \in \mathbb{N}_0$, with

$$\tilde{a}_\nu(t) := q^\nu a(q^\nu t), \quad \tilde{b}_\nu(t) := q^\nu b(q^\nu t),$$

and

$$\tilde{K}_{i,\nu}(t,s) := q^\nu K_{i,\nu}(t,s), \quad i = 1,2.$$

The kernels $K_{i,\nu}$ are those defined in Lemma 4.9.

This easily verified result leads to the VIDE analogue of Lemma 4.10.

Lemma 4.12. Assume that $a, b \in C(I)$, $K_1 \in C(D)$, and $K_2 \in C(D_\theta)$. Then the (unique) solution of the truncated (finite) system of VIDEs corresponding to (4.20),

$$\begin{aligned} z'_{M,\nu}(t) &= \tilde{a}_\nu(t) z_{M,\nu}(t) + \tilde{b}_\nu(t) z_{M,\nu+1}(t) \\ &\quad + \int_0^t \big(\tilde{K}_{1,\nu}(t,s) z_{M,\nu}(s) + \tilde{K}_{2,\nu}(t,s) z_{M,\nu+1}(s)\big)\,\mathrm{d}s, \\ &\qquad \nu = 0, 1, \ldots, M-1, \end{aligned} \tag{4.21}$$

$$z'_{M,M}(t) = \tilde{a}_\nu(t) z_{M,M}(t) + \int_0^t \tilde{K}_{1,M}(t,s) z_{M,M}(s)\,\mathrm{d}s, \quad t \in I, \tag{4.22}$$

with $z_{M,\nu}(0) = y_0$, satisfies

$$\|z_\nu(t) - z_{M,\nu}(t)\|_\infty \le Cq^{\tilde{M}}, \quad \nu = 0, 1, \ldots, M, \quad \text{with} \quad \tilde{M} \ge M.$$

Proof. Setting $\varepsilon_{M,\nu} := z_\nu - z_{M,\nu}$, we have

$$\varepsilon'_{M,\nu}(t) = \tilde{a}_\nu(t)\varepsilon_{M,\nu}(t) + \int_0^t \tilde{K}_{1,\nu}(t,s)\varepsilon_{M,\nu}(s)\,\mathrm{d}s + \Psi_{M,\nu}(t), \quad t \in I, \tag{4.23}$$

with $\varepsilon_{M,\nu}(0) = 0$, for $\nu = M, M-1, \ldots, 0$. Here,

$$\Psi_{M,\nu}(t) := \begin{cases} \tilde{b}_M(t)z_{M+1}(t) + \int_0^t \tilde{K}_{2,M}(t,s)z_{M+1}(s)\,\mathrm{d}s, & \text{if } \nu = M, \\ \tilde{b}_\nu(t)\varepsilon_{M,\nu+1}(t) + \int_0^t \tilde{K}_{2,\nu}(t,s)\varepsilon_{M,\nu+1}(s)\,\mathrm{d}s, & \text{if } \nu < M. \end{cases}$$

Let $r_{1,\nu} = r_{1,\nu}(t,s)$ denote the (differential) resolvent kernel corresponding to the functions $\tilde{a}_\nu$ and $\tilde{K}_{1,\nu}$ in (4.20); that is, $r_{1,\nu}$ is defined by the (unique) solution of the (differential) resolvent equation

$$\frac{\partial r_{1,\nu}(t,s)}{\partial s} = -r_{1,\nu}(t,s)\tilde{a}_\nu(s) - \int_s^t r_{1,\nu}(t,z)\tilde{K}_{1,\nu}(z,s)\,\mathrm{d}z, \quad (t,s) \in D,$$

with $r_{1,\nu}(t,t) = 1,\ t \in I$. The solution of the initial-value problem (4.21) can then be written in the form

$$\varepsilon_{M,\nu}(t) = r_{1,\nu}(t,0)\varepsilon_{M,\nu}(0) + \int_0^t r_{1,\nu}(t,s)\Psi_{M,\nu}(s)\,\mathrm{d}s, \quad t \in I, \tag{4.24}$$

for $\nu = M, M-1, \ldots, 0$, where $\varepsilon_{M,\nu}(0) = 0$ for all ν.

Using the estimate

$$|\Psi_{M,M}(t)| \le \gamma_0 q^M + \gamma_1 q^{2M+1}, \quad t \in I,$$

for finite constants γ_1 (recall that $\tilde{K}_{2,\nu}(t,s) = q^\nu K_{2,\nu}(t,s)$, with $|K_{2,\nu}(t,s)| \le \bar{K}_2 q^{\nu+1}$), we derive for $\nu = M$ that

$$|\varepsilon_{M,M}(t)| \le C_0 q^M + C_1 q^{2M+1} =: Cq^M, \quad t \in I,$$

where $C = C(q,M) < \infty$ for $M \in \mathbb{N}_0$ and $q \in (0,1)$.

For $\nu < M$ the argument for bringing the proof to its conclusion is analogous to the one in the proof of Lemma 4.10, except that now we employ the representation (4.24) and the estimate for $\|\varepsilon_{M,M}\|_\infty$. Details are left to the reader. □

Remark. The (uniform) convergence results in Lemmas 4.10 and 4.12 allow us not only to deduce the existence of unique solutions to the delay problems (4.6)) and (4.12) but also to establish the global regularity results already alluded to: C^m-data imply that the solutions of the DVIE and the DVIDE lie, respectively, in $C^m(I)$ and $C^{m+1}(I)$. We also encourage the reader to verify that the above embedding techniques can be extended to nonlinear Volterra equations with proportional delays, or with more general *nonlinear vanishing delays* described in Section 5.7.

5. Collocation methods for pantograph-type VFIEs

5.1. Numerical analysis of pantograph-type equations: an overview

The systematic study of the theory and the numerical analysis of the pantograph DDE and its various generalizations began with the papers by Ockendon and Tayler (1971), Fox *et al.* (1971), and Kato and McLeod (1971). While the theory of such functional equations almost immediately received much attention (see, for example, Frederickson (1971), Kato (1972), Bélair (1981), Derfel (1990, 1991), Kuang and Feldstein (1990), Derfel and Molchanov (1990), Iserles (1993, 1997*a*, 1994*b*), Terjéki (1995), Iserles and Terjéki (1995), Derfel and Vogl (1996), Liu (1996*a*), Iserles and Liu (1997), Feldstein and Liu (1998)), numerical analysts were singularly inattentive to the challenges of their numerical analysis: the fundamental paper on the numerical solution of the pantograph DDE (and its formulation as a Volterra functional equation) by Fox *et al.* (1971) stood alone until the early 1990s, when Buhmann and Iserles (1992, 1993), Iserles (1993), and Buhmann, Iserles and Nørsett (1993) understood that this class of functional differential equations represents a rich source of deep mathematical problems, both for the 'pure' and the numerical analyst.

5.1.1. Numerical analysis of the pantograph DDE

In the contributions just mentioned, the focus was on the asymptotic properties of numerical approximations, by linear multistep and simple collocation methods, for the pantograph equation (4.1). The survey by Iserles (1994*a*) and the papers of Iserles (1994*c*, 1997*a*, 1997*b*), Y. Liu (1995*a*, 1995*b*, 1996*a*, 1996*b*, 1997), Liang and Liu (1996), Liang, Qiu and Liu (1996), Bellen *et al.* (1997), Carvalho and Cooke (1998), Koto (1999), Liang and Liu (1996), Bellen (2001), Liu and Clements (2002), and Guglielmi and Zennaro (2003) describe various extensions of these early stability results, both on uniform and (quasi-) geometric meshes. Compare also the monograph by Bellen and Zennaro (2003) for a survey of many of these results, and Brunner (2004*a*) for additional references.

Collocation methods and their (super-) convergence properties are considered in Buhmann *et al.* (1993) (for $u_h \in S_1^{(0)}(I_h)$ and $q = 1/2$), Brunner (1997*a*), Zhang (1998), Zhang and Brunner (1998), Takama, Muroya and Ishiwata (2000), and Brunner (2004*b*, Chapter 5). While these properties are now reasonably well understood, this is not true for the qualitative aspects of piecewise polynomial (and continuous Runge–Kutta) methods: as shown in, *e.g.*, Buhmann *et al.* (1993), the present understanding is still at a very primitive level (except possibly when $q = 1/2$).

5.1.2. Volterra functional equations with proportional delays

Fox *et al.* (1971, pp. 292–295) used the integrated form of the pantograph equation, *i.e.*, a Volterra functional integral equation, to analyse the error induced by a variant of the classical Lanczos τ-method. Collocation methods for Volterra integral and integro-differential equations with proportional delays were studied in detail in Brunner (1997*a*), Zhang (1998), Zhang and Brunner (1998) (for second-order Volterra functional integro-differential equations), Takama *et al.* (2000), Ishiwata (2000), Muroya, Ishiwata and Brunner (2002), and Bellen *et al.* (2002). In these papers the focus is on the attainable orders of global and local (super-) convergence in collocation solutions. See also the survey by Brunner (2004*a*).

As we shall see in Section 5.8 the analysis of the asymptotic behaviour of collocation solutions to pantograph-type Volterra integral and integro-differential equations is completely open.

5.2. Piecewise polynomial collocation methods

5.2.1. Discretization on uniform meshes: overlap

The collocation equations corresponding to the pantograph-type delay Volterra equations to be discussed in Sections 5.2–5.3 will contain the delay integral terms $(\mathcal{V}_\theta u_h)(t)$ and $(\mathcal{W}_\theta u_h)(t)$, where $\theta(t) = qt$ $(0 < q < 1)$ and $t = t_{n,i} := t_n + c_i h_n \in X_h \cap \sigma_n$. Thus, the structure of the difference equations corresponding to these collocation points will be governed by the location of the images of these collocation points,

$$\theta(t_{n,i}) = q(t_n + c_i h_n), \quad i = 1, \ldots, m.$$

For arbitrary non-uniform meshes these difference equations are obviously very complex. Therefore, in order to make the analysis tractable, we will for the present assume that the mesh I_h is *uniform*:

$$I_h := \{t_n := nh : \; n = 0, 1, \ldots, N; \; t_N = T\}.$$

For a uniform mesh I_h and $t = t_{n,i} := t_n + c_i h \in X_h$ we will write

$$\theta(t_{n,i}) = q(t_n + c_i h) =: h\{q_{n,i} + \gamma_{n,i}\} = t_{q_{n,i}} + \gamma_{n,i} h, \tag{5.1}$$

where

$$q_{n,i} := \lfloor q(n + c_i) \rfloor \in \mathbb{N}_0, \qquad \gamma_{n,i} := q(n + c_i) - q_{n,i} \in [0, 1).$$

For given collocation parameters with $0 < c_1 < \cdots < c_m \le 1$ and $q \in (0, 1)$ define

$$q^I := \lceil qc_1/(1 - q) \rceil, \qquad q^{II} := \lceil qc_m/(1 - q) \rceil. \tag{5.2}$$

Here, $\lfloor x \rfloor$ is the greatest integer not exceeding $x \in \mathbb{R}$, and $\lceil x \rceil$ denotes the least upper integer bound for x.

The validity of the following lemma – which characterizes the 'overlap' of the images of the collocation points under the mapping θ – is easily verified.

Lemma 5.1. Let $q \in (0,1)$ and $0 < c_1 < \cdots < c_m \leq 1$, and assume that I_h is a uniform mesh on $I := [0,T]$ with diameter $h := T/N$. Then:

(i) for $n = 0$ we have $q(t_n + c_i h) \in (t_n, t_{n+1})$ for $i = 1, \ldots, m$,

(ii) if $n \geq 1$ we have $q(t_n + c_i h) \in (t_n, t_{n+1})$ if and only if $n < q^I$,

(iii) $q(t_n + c_i h) \leq t_n$ for $i = 1, \ldots, m$ if and only if $q^{II} \leq n \leq N-1$.

This result reveals that the recursive computation of the collocation solution for functional equations with (vanishing) proportional delay consists in general of *three phases*.

Phase I. During this 'initial phase' we have *complete overlap* which is described by the values of n satisfying

$$0 \leq n < \left\lceil \frac{q}{1-q} c_1 \right\rceil =: q^I.$$

Phase II. The 'transition phase' corresponds to those n where *partial overlap* occurs; they are given by

$$q^I \leq n < \left\lceil \frac{q}{1-q} c_m \right\rceil =: q^{II}.$$

If this set of indices n is non-empty, there exists a $\nu_n \in \{1, \ldots, m-1\}$ so that

$$q(t_n + c_i h) \leq t_n \quad (i \leq \nu_n) \quad \text{and} \quad q(t_n + c_i h) > t_n \quad (i > \nu_n).$$

Phase III. In this 'pure delay phase', described by

$$q^{II} \leq n \leq N-1,$$

there is no longer any *overlap* of $\theta(t_{n,i})$ and $t_{n,i}$: we have

$$q(t_n + c_i h) \leq t_n \quad \text{for} \quad i = 1, \ldots, m.$$

More precisely, for such a value of n there exist integers $\nu_n \in \{1, \ldots, m\}$ and $q_n < n-1$ such that

$$q_{n,i} = q_n \quad (i \leq \nu_n) \quad \text{and} \quad q_{n,i} = q_{n+1} \quad (i > \nu_n).$$

Note that the integers q^I and q^{II} do not depend on the underlying mesh and are thus independent of N.

Illustration. We list a selection values of q^I and q^{II} for $m = 2$ and $m = 3$, corresponding to the

Gauss points:

$$m = 2: \quad c_1 = (3 - \sqrt{3})/6, \qquad c_2 = (3 + \sqrt{3})/6,$$
$$m = 3: \quad c_1 = (5 - \sqrt{15})/10, \quad c_2 = 1/2, \qquad c_3 = (5 + \sqrt{15})/10.$$

Radau II points:

$$m = 2: \quad c_1 = 1/3, \qquad c_2 = 1,$$
$$m = 3: \quad c_1 = (4 - \sqrt{6})/10, \quad c_2 = (4 + \sqrt{6})/10, \quad c_3 = 1.$$

Table 5.1. $m = 2$.

	Gauss points				Radau II points			
q	1/2	2/3	0.9	0.99	1/2	2/3	0.9	0.99
q^I	1	1	2	21	1	1	3	33
q^{II}	1	2	8	79	1	2	9	99

Table 5.2. $m = 3$.

	Gauss points				Radau II points			
q	1/2	2/3	0.9	0.99	1/2	2/3	0.9	0.99
q^I	1	1	2	12	1	1	2	16
q^{II}	1	2	8	88	1	2	9	99

5.3. Second-kind VIEs with proportional delays

5.3.1. The structure of the collocation equations

The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for the delay integral equation (4.6) is defined by

$$u_h(t) = g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in X_h, \tag{5.3}$$

with $\theta(t) = qt$ $(0 < q < 1)$ and X_h given by (3.2) $(h_n = h)$. The contribution of the delay term $(\mathcal{V}_\theta u_h)(t_{n,i})$ to the collocation equation (5.3) will depend on n and the location of the collocation parameters $\{c_i\}$: it follows from the definition of $q_{n,i}$ and $\gamma_{n,i}$ in (5.1) that

$$(\mathcal{V}_\theta u_h)(t_{n,i}) = \int_0^{t_{q_{n,i}}} K(t_{n,i}, s)u_h(s)\,\mathrm{d}s + h\int_0^{\gamma_{n,i}} K(t_{n,i}, t_{q_{n,i}} + sh)u_h(t_{q_{n,i}} + sh)\,\mathrm{d}s. \tag{5.4}$$

In order to obtain suitable computational forms of the collocation equation (5.3) (leading to systems of difference equations whose solution describes the collocation solution on I), we again express u_h on the subinterval σ_n by the local Lagrange representation,

$$u_h(t_n + vh_n) = \sum_{j=1}^{m} L_j(v)U_{n,j}, \quad v \in (0, 1], \quad 0 \le n \le N-1, \tag{5.5}$$

with

$$L_j(v) := \prod_{k\neq j}^{m} \frac{s - c_k}{c_j - c_k} \quad \text{and} \quad U_{n,j} := u_h(t_n + c_j h_n),$$

and where we have assumed that $h_n = h = T/N$ for $n = 0, 1, \ldots, N-1$. The above expression for $(\mathcal{V}_\theta u_h)(t_{n,i})$ now becomes

$$(\mathcal{V}_\theta u_h)(t_{n,i}) = h\sum_{\ell=0}^{q_{n,i}-1}\sum_{j=1}^{m}\Bigl(\int_0^1 K(t_{n,i}, t_\ell + sh)L_j(s)\,\mathrm{d}s\Bigr)U_{\ell,j} + h\sum_{j=1}^{m}\Bigl(\int_0^{\gamma_{n,i}} K(t_{n,i}, t_{q_{n,i}} + sh)L_j(s)\,\mathrm{d}s\Bigr)U_{q_{n,i},j} \tag{5.6}$$

($i = 1, \ldots, m$). Thus, the collocation equation (5.3) leads to a system of linear difference equations for the vectors $\mathbf{U}_n \in \mathbb{R}^m$ ($0 \le n \le N-1$) whose structure will change as we pass from Phase I of *complete overlap* ($0 \le n < q^I$) via Phase II of *partial overlap* ($q^I \le n < q^{II}$) to the *pure delay* Phase III ($q^{III} \le n \le N-1$). To make this more precise, we set $\mathbf{g}_n := (g(t_{n,1}), \ldots, g(t_{n,m}))^T \in \mathbb{R}^m$ and introduce the matrices

$$B_n := \Bigl(\int_0^{c_i} K_1(t_{n,i}, t_n + sh)L_j(s)\,\mathrm{d}s\Bigr)_{i,j=1}^{m}, \tag{5.7}$$

$$\text{[2pt]}B_{n,\ell} := \Bigl(\int_0^{1} K_1(t_{n,i}, t_\ell + sh)L_j(s)\,\mathrm{d}s\Bigr)_{i,j=1}^{m}, \quad \ell < n, \tag{5.8}$$

in $L(\mathbb{R}^m)$. These matrices correspond to the contribution to the difference equation due to the ‘classical’ (non-delay) Volterra operator $\mathcal{V}$ in (5.3). For the concise formulation of the terms in the difference equation describing

Phases I, II and III and corresponding to the delay operator $\mathcal{V}_\theta$, we introduce the following matrices in $L(\mathbb{R}^m)$.

Phase I.

$$B_n^I(q) := \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_n + sh) L_j(s)\,\mathrm{d}s \right)_{i,j=1}^m, \tag{5.9}$$

$$B_{n,\ell}^I(q) := \left(\int_0^{1} K_2(t_{n,i}, t_\ell + sh) L_j(s)\,\mathrm{d}s \right)_{i,j=1}^m, \quad \ell < n. \tag{5.10}$$

Here, $\mathbf{U}_n$ $(0 \le n < q^I)$ is given by the solution of

$$[I_m - h(B_n + B_n^I(q))]\mathbf{U}_n = \mathbf{g}_n + h \sum_{\ell=0}^{n-1} (B_{n,\ell} + B_{n,\ell}^I(q))\mathbf{U}_\ell. \tag{5.11}$$

Phase II.

$$B_n^{II}(q) := \mathrm{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B_n^I(q), \tag{5.12}$$

$$B_{n-1}^{II}(q) := \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{n-1} + sh) L_j(s)\,\mathrm{d}s \right)_{i,j=1}^m, \tag{5.13}$$

$$S_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B_{n,n-1}^I(q), \tag{5.14}$$

$$\hat{S}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1, \dots, 1}_{\nu_n}, 0, \dots, 0) B_{n-1}^{II}(q). \tag{5.15}$$

It is readily verified that now $\mathbf{U}_n$ $(q^I \le n < q^{II})$ is determined by the solution of

$$[I_m - h(B_n + B_n^{II}(q))]\mathbf{U}_n = \mathbf{g}_n + h \sum_{\ell=0}^{n-1} B_{n,\ell}\mathbf{U}_\ell + h \sum_{\ell=0}^{n-2} B_{n,\ell}^I(q)\mathbf{U}_\ell + h[\hat{S}_{n-1}^{II}(q) + S_{n-1}^{II}(q)]\mathbf{U}_{n-1}. \tag{5.16}$$

Phase III.

$$B_{q_n}^{III}(q) := \left(\int_0^{\gamma_{n,i}} K_2(t_{n,i}, t_{q_n} + sh) L_j(s)\,\mathrm{d}s \right)_{i,j=1}^m, \tag{5.17}$$

$$S_{q_n+1}^{III}(q) := \mathrm{diag}(\underbrace{0, \dots, 0}_{\nu_n}, 1, \dots, 1) B_{q_n+1}^{III}(q), \tag{5.18}$$

$$\hat{S}_{q_n}^{III}(q) := \mathrm{diag}(\underbrace{1, \dots, 1}_{\nu_n}, 0, \dots, 0) B_{q_n}^{III}(q). \tag{5.19}$$

Once we have reached this pure delay phase the vector $\mathbf{U}_n$ ($q^{II} \leq n \leq N-1$) is given by the solution of

$$[I_m - hB_n]\mathbf{U}_n = \mathbf{g}_n + h\sum_{\ell=0}^{n-1} B_{n,\ell}\mathbf{U}_\ell + h\sum_{\ell=0}^{q_n-1} B^I_{n,\ell}(q)\mathbf{U}_\ell \qquad (5.20)$$
$$+ h[\hat{S}^{III}_{q_n}(q) + B^I_{n,q_n}(q)]\mathbf{U}_{q_n} + hS^{III}_{q_n+1}(q)\mathbf{U}_{q_n+1}.$$

The (different) integers ν_n occurring in Phases II and III, and the integer $q_n < n-1$ in Phase III were defined following Lemma 5.1.

Example 5.1. Suppose that the second-kind delay VIE (4.4) is solved by collocation in $S_0^{(-1)}(I_h)$, with uniform mesh I_h and collocation points $X_h = \{t_n + c_1 h:\ 0 < c_1 \leq 1\ (0 \leq n \leq N-1)\}$. Since u_h is constant on each subinterval σ_n we write $y_{n+1} := u_h(t_n + vh)$ ($v \in (0,1]$).

For $m = 1$ we have $q^I = q^{II} = \lceil qc_1/(1-q)\rceil$. Hence, the collocation equation (with $\mathcal{V} = 0$) becomes

$$\left(1 - h\int_0^{\gamma_{n,1}} K(t_{n,1}, t_n + sh)\,\mathrm{d}s\right) y_{n+1}$$
$$= h\sum_{\ell=0}^{n-1}\left(\int_0^1 K(t_{n,1}, t_\ell + sh)\,\mathrm{d}s\right) y_{\ell+1} + g(t_{n,1})$$

when $0 \leq n < q^I$. For $q^I \leq n \leq N-1$ it reads

$$y_{n+1} = g(t_{n,1}) + h\sum_{\ell=0}^{q_{n,1}-1}\left(\int_0^1 K(t_{n,1}, t_\ell + sh)\,\mathrm{d}s\right) y_{\ell+1}$$
$$+ h\left(\int_0^{\gamma_{n,1}} K(t_{n,1}, t_{q_{n,1}+1} + sh)\,\mathrm{d}s\right) y_{q_{n,1}+1}.$$

Thus, if $K(t,s) \equiv b/q$ and $g(t) \equiv 1$ the collocation solution to the resulting DVIE

$$y(t) = 1 + \int_0^{qt} (b/q)y(s)\,\mathrm{d}s, \quad t \in I$$

(which is equivalent to the initial-value problem $y'(t) = by(qt)$, $y(0) = 1$) is determined by the solution of the difference equation

$$y_{n+1} = 1 + \frac{hb}{q}\sum_{\ell=0}^{q_{n,1}-1} y_{\ell+1} + \frac{hb}{q}\gamma_{n,1}y_{q_{n,1}+1}, \qquad (5.21)$$

where $q_{n,1} := \lfloor qc_1/(1-q)\rfloor$ and $\gamma_{n,1} := q(n + c_1) - q_{n,1}$.

We list, also for use in Example 5.2, a sample of values of $q_{n,1}$ and $\gamma_{n,1}$ for $c_1 = 1/2$, to provide some insight into the structure of the above difference equations when collocation is at the Gauss point $t_n + h/2$.

Table 5.3. $q = 1/2,\ c_1 = 1/2\ (q^I = q^{II} = 1)$.

n	0	1	2	3	4	5	6
$q_{n,1}$	0	0	1	1	2	2	3
$\gamma_{n,1}$	1/4	3/4	1/4	3/4	1/4	3/4	1/4

Table 5.4. $q = 0.9,\ c_1 = 1/2\ (q^I = q^{II} = 5)$.

n	0	1	2	3	4	5	6
$q_{n,1}$	0	1	2	3	4	4	5
$\gamma_{n,1}$	0.45	0.35	0.25	0.15	0.05	0.95	0.85

For Volterra functional equations governed by the special delay operator $\mathcal{W}_\theta$ (*cf.* (4.8)) we find – in complete analogy to the above – that $(\mathcal{W}_\theta u_h)(t_{n,i})$ is given by

$$\begin{aligned}(\mathcal{W}_\theta u_h)(t_{n,i}) = h \int_{\gamma_{n,i}}^1 & K(t_{n,i}, t_{q_{n,i}} + sh) u_h(t_{q_{n,i}} + sh)\,\mathrm{d}s \\ &+ \int_{t_{q_{n,i}+1}}^{t_n} K(t_{n,i}, s) u_h(s)\,\mathrm{d}s \\ &+ h \int_0^{c_i} K(t_{n,i}, t_n + sh) u_h(t_n + sh)\,\mathrm{d}s.\end{aligned} \tag{5.22}$$

This can be written as

$$\begin{aligned}(\mathcal{W}_\theta u_h)(t_{n,i}) = h \sum_{j=1}^m & \Big(\int_{\gamma_{n,i}}^1 K(t_{n,i}, t_{q_{n,i}} + sh) L_j(s)\,\mathrm{d}s \Big) U_{q_{n,i},j} \\ &+ h \sum_{\ell=q_{n,i}+1}^{n-1} \Big(\int_0^1 K(t_{n,i}, t_\ell + sh) L_j(s)\,\mathrm{d}s \Big) U_{\ell,j} \\ &+ h \sum_{j=1}^m \Big(\int_0^{c_i} K(t_{n,i}, t_n + sh) K L_j(s)\,\mathrm{d}s \Big) U_{n,j}.\end{aligned} \tag{5.23}$$

Hence, the collocation equation associated with $y = g + \mathcal{W}_\theta y$,

$$u_h(t) = g(t) + (\mathcal{W}_\theta u_h)(t), \quad t \in X_h,$$

leads to the following systems of linear algebraic equations for $\mathbf{U}_n \in \mathbb{R}^m$ describing the local representation (5.5) of $u_h \in S_{m-1}^{(-1)}(I_h$ (again with uniform I_h).

Phase I.

$$[I_m - h\bar{B}_n^I(q)]\mathbf{U}_n = \mathbf{g}_n, \quad 0 \le n < q^I, \tag{5.24}$$

with

$$\bar{B}_n^I(q) = \left(\int_{\gamma_{n,i}}^{c_i} K(t_{n,i}, t_n + sh)L_j(s)\,\mathrm{d}s\right)_{i,j=1}^m,$$

which is formally equivalent to $B_n + B_n^I(q)$ in (5.11) when $K_2 = -K_1 =: -K$.

Phase II.

$$[I_m - h\bar{B}_n^{II}(q)]\mathbf{U}_n = \mathbf{g}_n + h\bar{S}_{n-1}^{II}(q)\mathbf{U}_{n-1}, \quad q^I \le n < q^{II}, \tag{5.25}$$

where

$$\bar{B}_n^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)B_n + \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)\bar{B}_n^I(q),$$

and

$$\bar{S}_{n-1}^{II}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)\left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{n-1} + sh)L_j(s)\,\mathrm{d}s\right).$$

Phase III.

$$[I_m - hB_n]\mathbf{U}_n = \mathbf{g}_n + h[\bar{S}_{q_n}^{III}(q)\mathbf{U}_{q_n} + \sum_{\ell=q_n+1}^{n-1} B_{n,\ell}\mathbf{U}_\ell + S_{q_n+1}^{III}(q)\mathbf{U}_{q_n+1}], \tag{5.26}$$

with

$$\bar{S}_{q_n}^{III}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)\left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_n} + sh)L_j(s)\,\mathrm{d}s\right),$$

$$S_{q_n+1}^{III}(q) := \mathrm{diag}(\underbrace{1,\ldots,1}_{\nu_n},0,\ldots,0)B_{n,q_n+1}$$
$$+ \mathrm{diag}(\underbrace{0,\ldots,0}_{\nu_n},1,\ldots,1)\left(\int_{\gamma_{n,i}}^{1} K(t_{n,i}, t_{q_n+1} + sh)L_j(s)\,\mathrm{d}s\right).$$

The matrices $B_{n,\ell} \in L(\mathbb{R}^m)$ coincide with the ones in (5.8).

5.3.2. Optimal orders of convergence

Suppose that a given delay integral with (vanishing) proportional delay is solved by collocation in $S_{m-1}^{(-1)}(I_h)$, with the underlying mesh I_h being a *uniform* one, and assume that the functions defining the functional equation have arbitrarily high degree of regularity on their respective domains (implying, as we have seen in Theorem 4.4, that the exact solution has the same regularity). What can be said, as $h \to 0,\ N \to \infty$ $(Nh = T)$ about the optimal values of p and p^* in the estimates

$$\|y - u_h\|_\infty := \max\{|y(t) - u_h(t)| :\ t \in I\} \le C(q)h^p \tag{5.27}$$

and

$$\|y - u_h\|_{I_h,\infty} := \max\{|y(t) - u_h(t)| :\ t \in I_h \setminus \{0\}\} \le C(q)h^{p^*}? \tag{5.28}$$

Do higher values of p and p^* result for the *iterated* collocation solution u_h^{it} based on u_h?

It turns out that we have $p = m$ in the global estimate (5.27) if the set $\{c_i\}$ is arbitrary; that is, the results that hold for 'classical' VIEs of the second kind remain valid. However, the question regarding the optimal value of p^* in (5.28) (attainable order of *local superconvergence* on $I_h \setminus \{0\}$) has an answer that differs sharply from the one for non-delay VIEs. Moreover, we shall see that it is not yet known under what conditions on the collocation parameters $\{c_i\}$ the collocation solutions u_h to the *first-kind* delay VIE (5.27) converge uniformly on I to the exact solution y (see also Section 5.2.1).

Theorem 5.2. Consider the second-kind delay VIE (4.6) and assume that the given functions g, K_1, K_2 are at least m times continuously differentiable on their respective domains. Then, for all sufficiently small $h > 0$ (so that the difference equations (5.11), (5.16), (5.20) possess unique solutions), we have $p = m$ in (5.27) for arbitrary $\{c_i\}$ in X_h and for all delay functions $\theta(t) = qt$ with $0 < q < 1$. The constant $C(q)$ depends on the $\{c_i\}$ and on q but not on h.

The proof of this result is an adaptation of the one for classical second-kind VIEs and delay VIEs with non-vanishing delays: it consists in showing that in each of the Phases I–III the ℓ^1-norms of the vectors $\mathbf{U}_n$ defined by (5.11), (5.16), and (5.20) satisfy a discrete Gronwall inequality. An elementary induction argument then leads to the assertion, observing the local representation (5.5) of u_h and the fact that, by the assumption on the regularity of the given data, $y \in C^m(I)$. The detailed arguments can be found in Chapter 5 (Section 5.3) of Brunner (2004*b*); see also Zhang (1998) and Brunner and Zhang (1999) for related results.

We now turn to the question of global and local *superconvergence* on I and I_h, respectively: are there collocation parameters $\{c_i\}$ for which $p = m$ in (5.27) can be replaced by $p^* > m$, and how large can p^* become in (5.28)?

Assume that the solution of the second-kind delay VIE (4.6) is approximated by the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ and the corresponding *iterated collocation solution*,

$$u_h^{\mathrm{it}}(t) := g(t) + (\mathcal{V}u_h)(t) + (\mathcal{V}_\theta u_h)(t), \quad t \in I. \tag{5.29}$$

Recall that $u_h^{\mathrm{it}}(t) = u_h(t)$ whenever $t \in X_h$; in particular, if $c_m = 1$ then $t_n \in X_h$ $(n = 1, \ldots, N)$ and hence $u_h^{\mathrm{it}}(t_n) = u_h(t_n)$.

It is well known (see, *e.g.*, Brunner and van der Houwen (1986, Chapter 5)) that if $\mathcal{V}_\theta = 0$ (classical Volterra integral equation) and if the $\{c_i\}$ are the *Gauss* (*–Legendre*) *points* (given by the zeros of the shifted Legendre polynomial $P_m(2s-1)$), then we only attain $p^* = m$ in (5.28): local superconvergence of order $p^* = 2m$ is only possible if u_h is replaced by the iterated collocation solution u_h^{it}. For the general delay VIE (4.6) with $\theta(t) = qt$ $(0 < q < 1)$ this is no longer true (see Brunner (1997*a*), Takama *et al.* (2000), Muroya *et al.* (2002), Brunner (2004*a*)): for *arbitrary* $q \in (0, 1)$ we have $p^* < 2m$ when $m \geq 3$. The results of Theorems 5.3 and 5.5 have recently been established in Brunner and Hu (2003); see also Brunner (2004*a*, 2004*b*). Theorem 5.5 disproves a conjecture in Brunner (1997*a*) and Brunner, Hu and Lin (2001*a*) for $m > 2$.

Theorem 5.3. Let the collocation parameters $\{c_i\}$ satisfy the orthogonality condition

$$J_0 := \int_0^1 \prod_{i=1}^{m} (s - c_i)\,\mathrm{d}s = 0. \tag{5.30}$$

Then

$$\|y - u_h^{\mathrm{it}}\|_\infty \leq C(q)h^{m+1},$$

where u_h^{it} is the iterated collocation approximation (5.29) corresponding to the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ for (4.4). Here, $m+1$ cannot, in general, be replaced by $m+2$.

We note that the most prominent set of parameters $\{c_i\}$ satisfying the above orthogonality condition are the *Gauss points*.

Proof. We will sketch the principal steps leading to the above global superconvergence result by using an approach different from the one in Brunner and Hu (2003). For ease of notation we will do this for (4.4) with $\mathcal{V} = 0$, that is,

$$y(t) = g(t) + (\mathcal{V}_\theta y)(t), \quad t \in I.$$

In this case, the collocation error $e_h := y - u_h$ satisfies the integral equation

$$e_h(t) = \delta_h(t) + (\mathcal{V}_\theta e_h)(t), \quad t \in I, \tag{5.31}$$

where the defect δ_h vanishes on X_h and inherits (piecewise, on each subinterval σ_n) the regularity of g and K_2. Moreover, we have $e_h^{\mathrm{it}} := y - u_h^{\mathrm{it}} = e_h - \delta_h$. Hence, it follows from the solution representation (4.5) in Theorem 4.2 that

$$e_h^{\mathrm{it}}(t) = \sum_{j=1}^{\infty} \int_0^{q^j t} K_{2,j}(t,s)\delta_h(s)\,\mathrm{d}s, \quad t \in I. \tag{5.32}$$

Here, $K_{2,j}(t,s)$ denotes the jth iterated kernel of the given kernel $K_2(t,s)$ in $\mathcal{V}_\theta$ (*cf.* Theorem 4.2)), and the infinite series converges uniformly on I.

Now let $t = t_n + vh$, $v \in [0,1]$ be given and define, as in (5.1),

$$q_{j,n}(v) := \lfloor q^j(n+v) \rfloor, \qquad \gamma_{j,n}(v) := q^j(n+v) - q_{j,n}(v) \in [0,1), \quad j \in \mathbb{N}.$$

This allows us to write (5.32) in the form

$$e_h^{\mathrm{it}}(t) = \sum_{j=1}^{\infty} \Big(\int_0^{t_{q_{j,n}(v)}} K_{2,j}(t,s)\delta_h(s)\,\mathrm{d}s \tag{5.33}$$
$$+ h \int_0^{\gamma_{j,n}(v)} K_{2,j}(t, t_{q_{j,n}}(v) + sh)\delta_h(t_{q_{j,n}}(v) + sh)\,\mathrm{d}s \Big).$$

The assertion in Theorem 5.3 now follows from the following observations.

(i) Since $0 < q < 1$, we have $q_{j,n}(v) < N$ for all $n \le N-1$ and $v \in [0,1]$. Upon writing

$$\int_0^{t_{q_{j,n}}(v)} K_{2,j}(t,s)\delta_h(s)\,\mathrm{d}s = h \sum_{\ell=0}^{q_{j,n}(v)-1} \int_0^1 K_{2,j}(t, t_\ell + sh)\delta_h(t_\ell + sh)\,\mathrm{d}s,$$

we can again resort to the classical 'quadrature error argument' of Sections 3.4.2 and 3.4.3, consisting in replacing each of the above integrals by the sum of the interpolatory m-point quadrature approximation based on the points $\{t_\ell + c_i h\}$ and the corresponding quadrature error. As $\delta_h(t_\ell + c_i h) = 0$, and $Nh = T$, it follows that the absolute values of the integrals are bounded by $C_Q h^{m+1}$, because the orthogonality condition (5.30) implies that the quadrature formulas all possess a degree of precision of (at least) m.

(ii) The global convergence estimate given by Theorem 5.2 can be used in (5.31) to obtain the estimate

$$\|\delta_h\|_\infty \le (1 + \|\mathcal{V}_\theta\|) C(q) h^m,$$

where

$$\|\mathcal{V}_\theta\| := \max_{t \in D_\theta} \int_0^{\theta(t)} |K_2(t,s)|\,\mathrm{d}s.$$

(iii) The iterated kernels $K_{2,j}(t,s)$ satisfy

$$|K_{2,j}(t,s)| \le \frac{q^{j(j-1)/2}}{(j-1)!} T^{j-1} \bar{K}_\theta^j, \quad t \in D_\theta^{(j)}$$

(*cf.* Lemma 4.3), where

$$\bar{K}_\theta := \max_{(D_\theta)} |K_2(t,s)| \quad \text{and} \quad D_\theta^{(j)} := \{(t,s):\ 0 \le s \le q^j t,\ t \in I\}.$$

This result implies that the infinite series involving the second terms on the right-hand side of (5.33) converges uniformly and is $\mathcal{O}(h^{m+1})$. □

In contrast to this result (the analogue of the global superconvergence result for ODEs), collocation at the Gauss points no longer leads to *local superconvergence* of order $2m$ on I_h when $m \ge 3$, as the following theorem shows. (Observe that for $m = 2$ we have $2m = m+2$.) A first hint that this is so may be divined from the following result comparing the collocation solution for the special pantograph equation (4.2) with the iterated collocation solution (based on the same collocation parameters) for its integrated form,

$$y(t) = y_0 + \int_0^{qt} (b/q) y(s)\,\mathrm{d}s, \quad t \in I.$$

Theorem 5.4. Let the following conditions hold.

(a) $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution (with respect to X_h) to the integrated form of $y'(t) = by(qt)$, $y(0) = y_0$ ($b \ne 0$, $y_0 \ne 0$), and u_h^{it} denotes the corresponding iterated collocation solution.

(b) $v_h \in S_m^{(0)}(I_h)$ is the collocation solution (also with respect to X_h) to $y'(t) = by(qt)$, $y(0) = y_0$.

Then we have, for all $q \in (0,1)$,

$$u_h^{\text{it}}(t) \ne v_h(t) \quad \text{whenever} \quad t \in I_h \setminus \{0\}.$$

Remark. We remind the reader that for $q = 1$ we obtain $u_h^{\text{it}}(t) = v_h(t)$ for all $t \in I$: the 'indirect' collocation approximation u_h^{it} has the same (super-)convergence properties as the 'direct' one for the original ODE.

The proof of the following theorem is based on interpolatory projection techniques and can be found in Brunner and Hu (2003).

Theorem 5.5. Assume that g, K_1 and K_2 in (4.6) are at least $d \ge m+2$ times continuously differentiable on their respective domains. Let $u_h \in S_{m-1}^{(-1)}(I_h)$ be the collocation solution corresponding to the Gauss points $\{c_i\}$, and let the iterated collocation solution be defined by (5.29). Then for any $q \in (0,1)$ the order p^* in the local estimate

$$\|y(t) - u_h^{\text{it}}(t)\|_{h,\infty} \le C(q) h^{p^*}$$

cannot exceed $m+2$, regardless of the value of d. More precisely, the following is true.

(i) If $q = 1/2$ then

$$\|y - u_h^{\mathrm{it}}\|_{h,\infty} \le C(q) \begin{cases} h^{m+2}, & \text{if } m \text{ is even}, \\ h^{m+1}, & \text{if } m \text{ is odd}. \end{cases}$$

(ii) For $q \in (0,1) \setminus \{1/2\}$, we have

$$\|y - u_h^{\mathrm{it}}\|_{h,\infty} \le C(q) h^{m+1},$$

for all $m \ge 2$.

Remarks. **(1)** It is intuitively clear that the error constants $C(q)$ in the above order estimates will change their values dramatically as $q \to 1^-$. Thus it is a challenging problem to find this dependence on q explicitly: even insight obtained from a simple (linear) 'toy problem' would be valuable.

(2) For the more general second-kind DVIE (4.9), which we write now as

$$y(t) = g(t) + b(t)y(\theta(t)) + (\mathcal{W}_\theta y)(t), \quad t \in I, \quad \theta(t) = qt, \quad 0 < q < 1,$$

the existence of a unique collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ is no longer guaranteed in Phases I and II for all sufficiently small mesh diameters $h > 0$. This is due to the presence of the terms $b(t_{n,i})u_h(\theta(t_{n,i}))$ which, using the local representation of u_h on $\sigma_{q_{n,i}}$ (*cf.* (5.5)), assume the form $b(t_{n,i}) \sum_{j=1}^{m} L_j(\gamma_{n,i}) U_{q_{n,i},j}$ $(i = 1, \dots, m)$. Thus, the matrix describing the left-hand side of the difference equation (5.11) (Phase I) now has the form $I_m - \mathcal{B}_n - h(B_n + B_n^I(q))$, with

$$\mathcal{B}_n := \mathrm{diag}(b(t_{n,i})) \begin{pmatrix} L_j(\gamma_{n,i}) \\ (i,j = 1,\dots,m) \end{pmatrix},$$

and hence its inverse will no longer exist for all sufficiently small $h > 0$. (Compare also Liu (1995*b*), where this problem is studied for the case $\mathcal{W}_\theta = 0$, $m = 1$, and $c_1 = 1$.) The superconvergence analysis for this more general Volterra integral equation with proportional delay is yet to be established.

5.4. Proportional delay VIEs of the first kind

Turning to Volterra's first-kind VIE with proportional delay $\theta(t) = qt$ $(0 < q < 1)$ of 1897,

$$(\mathcal{W}_\theta y)(t) = g(t), \quad t \in I := [0,T], \tag{5.34}$$

the linear algebraic systems whose solutions $\mathbf{U}_n \in \mathbb{R}^m$ define the local representations (5.5) of its collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ can be obtained from (5.24)–(5.26): they are, respectively,

$$\bar{B}_n^{I}(q)\mathbf{U}_n = h^{-1}\mathbf{g}_n, \quad 0 \le n < q^I, \tag{5.35}$$

$$\bar{B}_n^{II}(q)\mathbf{U}_n = h^{-1}\mathbf{g}_n - \bar{S}_{n-1}^{II}(q)\mathbf{U}_{n-1}, \quad q^I \le n < q^{II}, \tag{5.36}$$

and

$$\begin{aligned} B_n\mathbf{U}_n = h^{-1}\mathbf{g}_n - [\bar{S}_{q_n}^{III}(q)\mathbf{U}_{q_n} \\ + \sum_{\ell=q_n+1}^{n-1} B_{n,\ell}\mathbf{U}_\ell + S_{q_n+1}^{III}(q)\mathbf{U}_{q_n+1}, \quad q^{II} \le n \le N-1. \end{aligned} \tag{5.37}$$

Example 5.2. Consider the first-kind delay VIE (5.34), and assume that its collocation solution is to be in the collocation space of Example 5.1. Thus, for $0 \le n < q^I$ the collocation equation at $t = t_{n,1} = t_n + c_1 h$ may be written in the form

$$\left(\int_{\gamma_{n,1}}^{c_1} K(t_{n,1}, t_n + sh)\,\mathrm{d}s\right) y_{n+1} = h^{-1} g(t_{n,1}).$$

If $n \ge q^I = q^{II}$ (see Example 5.1) we have, by (5.22),

$$(\mathcal{W}_\theta u_h)(t_{n,1}) = \int_{qt_{n,1}}^{t_n} K(t_{n,1}, s) u_h(s)\,\mathrm{d}s + \int_{t_n}^{t_{n,1}} K(t_{n,1}, s) u_h(s)\,\mathrm{d}s,$$

and this can be written as

$$\begin{aligned} &\left(\int_0^{c_1} K(t_{n,1}, t_n + sh)\,\mathrm{d}s\right) y_{n+1} \\ &\quad = h^{-1} g(t_{n,1}) - \left(\int_{\gamma_{n,1}}^{1} K(t_{n,1}, t_{q_{n,1}} + sh)\,\mathrm{d}s\right) y_{q_{n,1}+1} \\ &\qquad - \sum_{\ell=q_{n,1}+1}^{n-1} \left(\int_0^1 K(t_{n,1}, t_\ell + sh)\,\mathrm{d}s\right) y_{\ell+1}. \end{aligned}$$

Setting

$$A_n := \int_0^{c_1} K(t_{n,1}, t_n + sh)\,\mathrm{d}s,$$

$$B_{n,\ell} := \int_0^1 K(t_{n,1}, t_\ell + sh)\,\mathrm{d}s \ (q_{n,1} + 1 \le \ell \le n-1),$$

and

$$C_{n,q_n} := \int_{\gamma_{n,1}}^{1} K(t_{n,1}, t_{q_{n,1}} + sh)\,\mathrm{d}s,$$

with $q_{n,1} := \lfloor q(n + c_1) \rfloor$ and $\gamma_{n,1} := q(n + c_1) - q_{n,1}$, the above difference equation for $\{y_{n+1}\}$ becomes

$$A_n y_{n+1} + \sum_{\ell=q_{n,1}+1}^{n-1} B_{n,\ell} y_{\ell+1} + C_{n,q_n} y_{q_{n,1}+1} = h^{-1} g(t_{n,1}). \tag{5.38}$$

If $K(t,s) \equiv 1$ the delay integral equation (5.34) reduces to

$$\int_{qt}^{t} y(s)\,\mathrm{d}s = g(t), \quad t \in I, \quad g(0) = 0,$$

and this is equivalent to the functional equation

$$y(t) - qy(qt) = g'(t), \quad t \in I.$$

The corresponding collocation solution $u_h \in S_0^{(-1)}(I_h)$ is thus determined by the solution of the difference equation

$$c_1 y_{n+1} + \sum_{\ell=q_{n,1}+1}^{n-1} y_{\ell+1} + (1-\gamma_{n,1}) y_{q_{n,1}+1} = h^{-1} g(t_n + c_1 h), \quad n \geq 0. \tag{5.39}$$

In order to obtain the difference equation corresponding to collocation at the Gauss points ($c_1 = 1/2$) recall Tables 5.3 and 5.4 of Example 5.1 for the values of $q_{n,1}$ and $\gamma_{n,1}$.

What can be said about the attainable orders of global and local (super-) convergence: what do the analogues of Theorems 5.2, 5.3 and 5.4 look like? As we have briefly mentioned before, the analysis leading to answers for these questions remains open. We only know from numerical evidence that the condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} \leq 1$$

(which guarantees uniform convergence when $q = 0$ in $\mathcal{W}_\theta$ with $\theta(t) = qt$) is no longer sufficient for this to be true. In particular, it is not yet known for which values of $c_1 \in (0,1]$ the solution of the simple difference equation (5.38) remains uniformly bounded as $N \to \infty$ ($h \to 0$, $Nh = T$) when $q \in (0,1)$.

5.5. Volterra integro-differential equations with proportional delays

The collocation solution $u_h \in S_m^{(0)}(I_h)$ for the DVIDE (4.12) is determined by

$$u_h'(t) = a(t)u_h(t) + b(t)u_h(\theta(t)) + (\mathcal{W}_\theta u_h)(t), \quad t \in X_h, \tag{5.40}$$

with initial condition $u_h(0) = y_0$. Since the collocation space is now a subspace of $C(I)$ the system of difference equations arising in the computation of u_h has a somewhat more complex structure than the ones we encountered in the previous section. To be somewhat more precise, the local representation of $u_h \in S_m^{(0)}(I_h)$ is now

$$u_h(t_n + vh) = y_n + h \sum_{j=1}^{m} \beta_j(v) Y_{n,j}, \quad v \in [0,1], \quad 0 \le n \le N-1, \tag{5.41}$$

where $y_n := u_h(t_n)$, $Y_{n,j} := u_h'(t_{n,j})$, and $\beta_j(v) := \int_0^1 L_j(s)\,\mathrm{d}s$. Hence, the key ingredients in the derivation of the difference equations are essentially the same as in Section 5.2, that is, the terms of $(\mathcal{W}_\theta u_h)(t_{n,i})$ corresponding to Phases I, II, and III and containing the vectors $\mathbf{Y}_n$ are described by matrices similar to those in (5.24)–(5.26), except that instead of $L_j(s)$ their integrands contain the integrated Lagrange polynomials $\beta_j(s)$. However, there are now also additional terms reflecting the continuity constraint of u_h at the interior mesh points $t_1, \ldots, t_{N-1}$. We leave the details to the reader; compare also Chapter 5 in Brunner (2004*b*). Instead, we present an illustration from which the difference equations for arbitrary $m \ge 2$ can readily be deduced.

Example 5.3. Consider (5.40) and suppose that $u_h \in S_1^{(0)}(I_h)$ $(m = 1)$, with $0 < c_1 \le 1$. The collocation equation defining this collocation solution u_h,

$$u_h'(t_{n,1}) = a(t_{n,1})u_h(t_{n,1}) + b(t_{n,1})u_h(qt_{n,1}) + (\mathcal{W}_\theta u_h)(t_{n,1}),$$

where, by (5.41), $u_h(t_n + vh) = y_n + hvY_{n,1}$ $(v \in [0,1])$, can be written as

$$Y_{n,1} = a(t_{n,1})\{y_n + hc_1Y_{n,1}\} + b(t_{n,1})\{y_{q_{n,1}} + h\gamma_{n,1}Y_{q_{n,1}}\} + (\mathcal{W}_\theta u_h)(t_{n,1}),$$

with

$$\begin{aligned}(\mathcal{W}_\theta u_h)(t_{n,1}) = &\int_{t_{q_{n,1}}}^{t_{q_{n,1}+1}} K(t_{n,1}, s)u_h(s)\,\mathrm{d}s + \int_{t_{q_{n,1}+1}}^{t_n} K(t_{n,1}, s)u_h(s)\,\mathrm{d}s \\ &+ h \int_0^{c_1} K(t_{n,1}, t_n + sh)\{y_n + hsY_{n,1}\}\,\mathrm{d}s.\end{aligned}$$

Hence, the resulting difference equation is

$$\begin{aligned}&\left(1 - ha(t_{n,1})c_1 - h^2 \int_0^{c_1} K(t_{n,1}, t_n + sh)s\,\mathrm{d}s\right) Y_{n,1} \\ &\quad = \left[hb(t_{n,1})\gamma_{n,1} + h^2\left(\int_{\gamma_{n,1}}^1 K(t_{n,1}, t_{q_{n,1}} + sh)s\,\mathrm{d}s\right)\right] Y_{q_{n,1}} \\ &\qquad + h^2 \sum_{\ell=q_{n,1}+1}^{n-1} \left(\int_0^1 K(t_{n,1}, t_\ell + sh)s\,\mathrm{d}s\right) Y_{\ell,1} + \rho_n,\end{aligned} \tag{5.42}$$

where

$$\rho_n := a(t_{n,1})y_n + \left(b(t_{n,1}) + h\int_{\gamma_{n,1}}^{1} K(t_{n,1}, t_{q_{n,1}} + sh)\,\mathrm{d}s\right) y_{q_{n,1}}$$
$$+ h \sum_{\ell=q_{n,1}+1}^{n-1} \left(\int_0^1 K(t_{n,1}, t_\ell + sh)\,\mathrm{d}s\right) y_\ell$$
$$+ h\left(\int_0^{c_1} K(t_{n,1}, t_n + sh)\,\mathrm{d}s\right) y_n.$$

The values of $q_{n,1}$ and $\gamma_{n,1}$ can be found in Example 5.1. Observe also that the above difference equation (5.42) can be reformulated as a difference equation for $\{y_n\}$, by setting $Y_{n,1} = (y_{n+1} - y_n)/h$ (*cf.* (5.41) with $m = 1$ and $v = 1$).

The problem of asymptotic stability for u_h in special case corresponding to $\mathcal{W} = 0$ and constant coefficients a, b, *i.e.*, the pantograph equation (4.1), was studied in Buhmann *et al.* (1993) and Iserles (1994*a*) for $q = 1/2$. For pantograph-type VIDEs the problem is completely open (see also Section 5.7).

5.5.1. Optimal convergence estimates

The first theorem shows that the classical global order of convergence also remains valid for delay VIDEs with vanishing proportional delay. The proof is again based on a 'Phase I–III' Gronwall-type inductive argument; the details can be found in Brunner (2004*b*, Section 5.5). See also Zhang (1998) and Zhang and Brunner (1998) for some related results.

Theorem 5.6. Consider the delay VIDE (4.12) (which includes (4.14) and the pantograph equation (4.1) as special cases) and assume that the given functions a, b, K_1 and K_2 are at least m times continuously differentiable on their respective domains. Then, for any $\theta(t) = qt$ with $q \in (0, 1)$ and for all sufficiently small $h > 0$, the unique collocation solution $u_h \in S_m^{(0)}(I_h)$ given by (5.40) satisfies

$$\|y^{\nu)} - u_h^{(\nu)}\|_\infty := \sup\{|y^{(\nu)}(t) - u_h^{(\nu)}(t)| :\ t \in I\} \le C_\nu(q)h^m, \quad \nu = 0, 1,$$

and this holds for any set $\{c_i\}$ defining the collocation points X_h. The constants $C_\nu(q)$ depend on the $\{c_i\}$ and on q but not on h.

As for delay VIEs of the second kind with proportional delays, *local superconvergence results* for analogous delay VIDEs differ sharply from the classical results. However, the *global order* $p = m + 1$ is possible for the collocation solution $u_h \in S_m^{(0)}(I_h)$. We summarize this results, and a conjecture, below. Note, incidentally, that these delay VIDEs include the pantograph equation and its counterpart with variable coefficients a and b as special cases.

Theorem 5.7. Let $u_h \in S_m^{(0)}(I_h)$ be the collocation solution to the proportional delay VIDE (4.12). If the given functions possess continuous derivatives of at least order $m+1$ on their respective domains, and if the $\{c_i\}$ satisfy the orthogonality condition (5.30) of Theorem 5.3, then

$$\|y - u_h\|_\infty \le C(q)h^{m+1}$$

holds for all $q \in (0,1)$.

Proof. We write the error equation for (5.40),

$$e_h'(t) = a(t)e_h(t) + b(t)e_h(qt) + \delta_h(t) + (\mathcal{V}e_h)(t) + (\mathcal{V}_\theta e_h)(t),$$
$$t \in I, \quad e_h(0) = 0,$$

in integrated form. The analysis in Brunner and Hu (2003) (or, if $a(t) \equiv 0$, $\mathcal{V} = 0$, the one employed in the proof of Theorem 5.3) can then be applied to the resulting delay integral equation for e_h. We leave the details to the reader. □

Conjecture 5.8. The order of local superconvergence of $u_h \in S_m^{(0)}(I_h)$ on I_h cannot exceed $p^* = m+2$. If the $\{c_i\}$ defining the collocation points X_h are the Gauss points, then we have $p^* = m+2$ for any $q \in (0,1)$ and all $m \ge 2$. The same is true for the general pantograph equation corresponding to $\mathcal{W}_\theta = 0$ in (4.14).

5.6. Collocation on geometric meshes

The special form of the delay function $\theta(t) = qt$ $(0 < q < 1)$ suggests that u_h^{it} might possibly attain the classical optimal order of superconvergence $p^* = 2m$ on I_h if I_h is a suitable *geometric mesh* and if collocation is at the Gauss points. That this is (almost) so was verified in the paper by Brunner *et al.* (2001*a*). We briefly describe this result and sketch its proof.

Assume that I_h is a *geometric mesh* defined by

$$I_h := \{t_n : \ t_n = \gamma^{N-n}T, \ n = 0,1,\ldots,N; \ \gamma \in (0,1)\}. \qquad (5.43)$$

The mesh parameter γ will depend, as is made precise below, on N (but not on n), on q, and on m. This mesh possesses the following obvious properties.

(i) $h_n := t_{n+1} - t_n = \gamma^{N-n-1}(1-\gamma)T \quad (n = 0,1,\ldots,N-1)$.

(ii) $\max_{(n)} h_n = h_{N-1} = (1-\gamma)T$ (for *any* $N \in \mathbb{N}$). Hence, $\gamma = \gamma(N)$ will have to be chosen so that $\gamma \to 1^-$, as $N \to \infty$, for all $q \in (0,1)$.

Let $\rho \in \mathbb{N}$ be defined by

$$\rho := \left\lfloor \frac{\ln(q)}{\ln(1 - \frac{2m\ln(N)}{(m+1)N})} \right\rfloor. \tag{5.44}$$

It is the largest integer for which

$$q^{1/\rho} \leq 1 - \frac{2m \cdot \ln(N)}{(m+1)N}.$$

Theorem 5.9 will reveal the motivation for introducing this integer ρ. Observe that for given (fixed) $q \in (0,1)$ and $m \geq 1$, we have $\rho > 1$ for all sufficiently large N. This is true because

$$1 - \frac{2m \cdot \ln(N)}{(m+1)N} \longrightarrow 1^-, \quad \text{as} \quad N \to \infty,$$

for any $m \in \mathbb{N}$.

Theorem 5.9. Let the following be satisfied.

(a) $g \in C^{2m}(I),\ K_1 \in C^{2m}(D),\ K_2 \in C^{2m}(D_\theta)$.
(b) I_h is the geometric mesh described by (5.43) and (5.44), with $\gamma = q^{1/\rho}$.
(c) $u_h \in S_{m-1}^{(-1)}(I_h)$ is the collocation solution to the delay VIE (4.6), with the $\{c_i\}$ given by the Gauss points, and u_h^{it} denotes the corresponding iterated collocation solution.

Then, for all sufficiently large N, the resulting local order of convergence of u_h^{it} is given by

$$\max_{t \in I_h \setminus \{0\}} |y(t) - u_h^{\text{it}}(t)| \leq C(q) N^{-(2m - \varepsilon_N)},$$

where

$$\varepsilon_N := \log_N \left(\frac{(2m \cdot \ln(N))^{2m}}{(2m+1)(m+1)^{2m}} \right)$$

satisfies

$$\lim_{N \to \infty} \varepsilon_N = 0.$$

Proof. Since the proof is technically quite complex (using interpolatory projection techniques), we will only exhibit one of its key ingredients. □

Lemma 5.10. Let I_h be the geometric mesh (5.43), (5.44), with $\gamma = q^{1/\rho}$. Then:

(i) $h_0 \leq C N^{-2m/(m+1)}$;
(ii) $\sum_{n=1}^{N-1} h_n^{2m+1} \leq C N^{-(2m - \varepsilon_N)}$;
(iii) for $\rho + 1 \leq n \leq N$ we have $q t_n = t_{n-\rho} \in I_h \setminus \{0\}$.

Note, incidentally, that proposition (iii) may be viewed as generalized θ-invariance for this geometric mesh I_h.

Remarks. **(1)** Geometric meshes similar to the ones employed here were introduced by Hu (1998) for piecewise polynomial collocation methods applied to VIDEs with weakly singular kernels, to obtain local superconvergence of the collocation solution on I_h.

(2) It is clear that analogous superconvergence results can be derived for collocation solutions in $S_m^{(0)}(I_h)$, with suitable geometric mesh I_h, for the VIDE (4.12) with proportional delays. However, since this has not yet been worked out in detail, the reader is invited to take up the challenge.

5.7. Equations with nonlinear vanishing delays

Suppose that the linear delay function $\theta(t) = qt$ $(0 < q < 1)$ is replaced by a *nonlinear* function θ satisfying the following conditions:

(ND1) $\theta \in C^1(I)$, with $\theta(0) = 0$ and $\theta(t) < t$ for $t > 0$;
(ND2) $\min_{t \in I} \theta'(t) = q_0 > 0$.

The (linear) proportional delay function $\theta(t) = qt$ $(0 < q < 1)$ of course satisfies (ND1) and (ND2), with $\theta'(t) = q =: q_0$ for all t. Similar nonlinear vanishing delays were considered by Denisov and Korovin (1992) and Denisov and Lorenzi (1995); see also Bellen *et al.* (2002).

While it seems clear from the foregoing convergence analyses that the (super-) convergence results for collocation solutions remain valid for (3.8) and (3.45) with such nonlinear vanishing delays, the details have yet to be worked out.

5.8. Open problems

Our previous discussion of collocation methods has made it clear that even for delay VFIEs and VFIDEs with the simple linear lag function $\theta(t) = qt$ $(0 < q < 1;\ t \in [0, T])$ we have a long way to go to understand the dynamics of collocation solutions. We list below a selection of open problems whose solution would significantly advance our understanding of such functional equations.

(1) Superconvergence analysis of collocation solutions in $S_m^{(0)}(I_h)$, with uniform mesh I_h, for *nonlinear* Volterra functional integro-differential equations of Hammerstein type,

$$y'(t) = f(y(t), y(qt), y'(qt)) + \int_{qt}^{t} k(t-s)G(y(s), y'(s))\,\mathrm{d}s, \quad 0 < q < 1,$$

in particular if f has one of the (Riccati, or rational) forms considered in Iserles (1994*b*).

(2) Long-time integration of pantograph-type functional equations: how do the error constants associated with the collocation solutions for DVIEs and DVIDEs depend on q and grow as $q \uparrow 1^-$]?

(3) For which continuous convolution kernels k_1 and k_2 in

$$y(t) = 1 + \int_0^t k_1(t-s)y(s)\,\mathrm{d}s + \int_0^{qt} k_2(t-s)y(s)\,\mathrm{d}s, \quad t \geq 0, \tag{5.45}$$

does

$$\lim_{t\to\infty} y(t) = 0$$

hold?

The answer to the analogous question for

$$y'(t) = ay(t) + by(qt) + \int_0^t k_1(t-s)y(s)\,\mathrm{d}s + \int_0^{qt} k_2(t-s)y(s)\,\mathrm{d}s, \quad t \geq 0, \tag{5.46}$$

where $y(0) = y_0 \neq 0$, is also open. In particular, for which continuous convolution kernels k do we obtain asymptotic stability in (5.43) and (5.44) when the sum of the two integral operators is replaced by the Volterra operator corresponding to $\mathcal{W}_\theta$,

$$(\mathcal{W}_q\phi)(t) := \int_{qt}^t k(t-s)\phi(s)\,\mathrm{d}s?$$

(4) Suppose that the given functions in (6.1) and (6.2) are such that the solutions of these pantograph-type Volterra equations are asymptotically stable. For which collocation parameters $\{c_i\}$ do the corresponding collocation solutions possess the same asymptotic behaviour?

6. Summary and outlook

6.1. Delay VIEs with weakly singular kernels

Owing to limitation of space we can only briefly touch upon the convergence properties of collocation solutions to delay VIEs of the form

$$y(t) = g(t) + (\mathcal{V}_\alpha y)(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in (t_0, T], \tag{6.1}$$

where the Volterra integral operators are based on kernels containing a weak (*i.e.*, unbounded but integrable) singularity:

$$(\mathcal{V}_\alpha y)(t) := \int_{t_0}^t (t-s)^{-\alpha} K_1(t,s)y(s)\,\mathrm{d}s,$$

$$(\mathcal{V}_{\theta,\alpha} y)(t) := \int_{t_0}^{\theta(t)} (t-s)^{-\alpha} K_2(t,s)y(s)\,\mathrm{d}s,$$

with $0 < \alpha < 1$, smooth K_i, and $K_1(t,t) \neq 0$ on I. The generalization of $\mathcal{W}_\theta$ in (3.22) is thus given by

$$(\mathcal{W}_{\theta,\alpha}y)(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha} K(t,s) y(s)\,\mathrm{d}s,$$

with $K(t,t) \neq 0$ on I.

It follows from the theory of classical (non-delay) VIEs with weakly singular kernels (Brunner, Pedas and Vainikko 1999) that on *uniform meshes* the collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ to (6.1) has global order of convergence of order $p = 1-\alpha$, regardless of the regularity of the given functions or the degree of the approximating piecewise polynomial u_h. This is due to the low regularity of y at $t = t_0^+$: its first derivative behaves like $C(t-t_0)^{-\alpha}$ near $t = t_0^+$. An analogous result holds for VIDEs with weakly singular kernels (see Brunner, Pedas and Vainikko (2001*b*)): here, it is y'' that is similarly unbounded at $t = t_0^+$.

The optimal order of global convergence, $p = m$, can only be restored if the mesh I_h is suitably *graded*, *i.e.*, if it is given by

$$I_h = \left\{ t_n := t_0 + \left(\frac{n}{N}\right)^r (T - t_0) :\ n = 0, 1, \ldots, N \right\},$$

with grading exponent $r \geq m/(1-\alpha)$ in the case of (6.1) with $\mathcal{V}_{\theta,\alpha} = 0$. For the analogous VIDE we must have $r \geq (m+1-\alpha)/(2-\alpha)$ (see Brunner (1985), Brunner, Pedas and Vainikko (1999, 2001)).

When solving weakly singular VIEs with *non-vanishing delays*, the mesh I_h will obviously be the constrained mesh (3.3) where the local meshes are now graded ones:

$$I_h^{(\mu)} := \left\{ t_n^{(\mu)} := \xi_\mu + \left(\frac{n}{N}\right)^{r_\mu} (\xi_{\mu+1} - \xi_\mu) :\ n = 0, 1, \ldots, N \right\},$$

with (local) grading exponents $r_\mu > 1$ depending on α and μ. Note that we have $\xi_{\mu+1} - \xi_\mu = \tau(\xi_{\mu+1}) \geq \tau_0 > 0$.

Lemma 6.1. Let I_h be a θ-invariant mesh defined by (3.3) and (3.4), and assume that the first local mesh $I_h^{(0)}$ is optimally graded, that is,

$$I_h^{(0)} := \left\{ t_n^{(0)} = \left(\frac{n}{N}\right)^{r_0} (\xi_1 - t_0) :\ n = 0, 1, \ldots, N \right\},$$

with $r_0 = m/(1-\alpha)$.

(i) If the lag function θ is linear, then the optimal grading is inherited by the macro-meshes $I_h^{(1)}, \ldots, I_h^{(M)}$, with $r_\mu = r_0$ for all μ.

(ii) If θ is nonlinear, the (optimal) grading is lost for $I_h^{(\mu)}$ ($\mu \geq 1$).

This result tells us that for the weakly singular Volterra integral or integro-differential equations (2.30) and (2.31) with non-vanishing delays, collocation on θ-invariant meshes I_h with optimally graded initial mesh $I_h^{(0)}$ will exhibit the classical optimal global and local convergence orders if the *delay function* θ is *linear*. For *nonlinear* θ, this will no longer be true.

Similar (positive and negative) results hold for piecewise polynomial collocation solutions applied to *neutral* VFIDEs with weakly singular kernels,

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_0 y(t) - (\mathcal{T}_{\theta,\alpha} y)(t)) = F(t, y(t), y(\theta(t)), y'(\theta(t))), \tag{6.2}$$

with $\mathcal{T}_{\theta,\alpha}$ denoting one of the Volterra delay operators $\mathcal{V}_{\theta,\alpha}$ or $\mathcal{W}_{\theta,\alpha}$ $(0 < \alpha < 1)$, when $a_0 = 1$. An alternative numerical approach to such functional integro-differential equations can be found in Ito and Turi (1991): it is based on the semigroup framework underlying functional equations of this kind.

For VFIDEs of the first kind, corresponding to $a_0 = 0$ in (6.2), *e.g.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}((\mathcal{W}_{\theta,\alpha} y)(t)) = f(t), \quad t \in (t_0, T],$$

the convergence analysis of collocation solutions is much more difficult, and many questions remain to be answered. This is not so surprising when we recall that collocation solutions in $S_{m-1}^{(-1)}(I_h)$ for classical first-kind VIEs,

$$(\mathcal{V}y)(t) = g(t), \quad t \in I := [0, T], \quad g(0) = 0,$$

converge to y uniformly on I only if the collocation parameters $\{c_i\} \subset (0, 1]$ satisfy the stability condition

$$-1 \leq \rho_m := (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1$$

(see Brunner and van der Houwen (1986) and Kauthen and Brunner (1997)). For the weakly singular version,

$$(\mathcal{V}_\alpha y)(t) = g(t), \quad t \in I, \quad 0 < \alpha < 1,$$

it is not known for which sets $\{c_i\}$ the collocation solution is convergent (see Brunner (1999*b*)). We note that Ito and Turi (1991) use their semigroup-based method to solve NVFIDEs of the form (6.2) however, the question regarding the attainable order of convergence on graded meshes remains open.

6.2. Integral-algebraic Volterra equations with non-vanishing delays

The numerical solution of differential-algebraic equations (DAEs) with constant delays by Runge–Kutta and collocation methods is studied in Ascher and Petzold (1995) and in Hauber (1997). However, as we mentioned in the discussion following equation (1.10), it is not clear how to obtain a *numerically properly formulated* form of a delay DAE, or of an analogous system of integral-algebraic or integro-differential-algebraic Volterra equations with delay arguments. This is an important issue for the understanding of the quantitative and – especially – the dynamical properties of collocation solutions to functional equations like (1.10). Although some partial answers are known for index-1 IAEs and IDAEs (see Chapter 8 in Brunner (2004*b*)), the numerical analysis of DDAEs and analogous delay Volterra IAEs of higher index is much more challenging. As März (2002*a*) has shown, the key to our understanding may possibly be found in a suitable reformulation of integral-algebraic equations as abstract (infinite-dimensional) DAEs, to which the elegant analysis of März (1992, 2002*b*) can be extended.

6.3. VFIEs with state-dependent delays

The numerical analysis of DDEs with state-dependent delays is now well understood. The papers by Feldstein and Neves (1984), Neves and Thompson (1992), Hartung and Turi (1995), Hartung, Herdman and Turi (1997), Györi, Hartung and Turi (1998), and the monograph by Bellen and Zennaro (2003) convey a fairly complete picture of its state of the arts.

For Volterra functional integro-differential equations with state-dependent delays we have the substantial work by Tavernini (1978) on general one-step methods. Cahlon and Nachman (1985) and Cahlon (1992) deal with a class of numerical methods for solving analogous Volterra functional integral equations. However, except for the results in Cryer and Tavernini (1972) (Euler's method may be viewed as a simple collocation method) the general (super-) convergence analysis for piecewise collocation methods is still outstanding. For example, we do not know if the collocation solution $u_h \in S_1^{(0)}(I_h)$ for DVIDEs of the form

$$y'(t) = g(t) + \int_{t-\tau(y(t))}^{t} k(t-s)G(y(s))\,\mathrm{d}s$$

(*i.e.*, the analogue of (1.8)) exhibits $\mathcal{O}(h^2)$-superconvergence if collocation is based on the Gauss point $c_1 = 1/2$. We are similarly ignorant about the optimal order of convergence on I_h for u_h^{it} corresponding to the collocation solution $u_h \in S_0^{(-1)}(I_h)$ for Bélair's state-dependent delay integral equation (1.8).

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